Communication Between Information Systems Using Fuzzy Rough Sets

E. C. C. Tsang, Changzhong Wang, Degang Chen, Congxin Wu, and Qinghua Hu

***Abstract*—Communication between information systems is a basic problem in granular computing, and the concept of homomorphism is a useful mathematical tool to study this problem. In this paper, some properties of communication between information systems based on fuzzy rough sets are investigated. The concepts of fuzzy relation mappings between universes are first proposed in order to construct a fuzzy relation of one universe according to the given fuzzy relation on the other universe. The main properties of the mappings are studied. The notions of homomorphism of information systems based on fuzzy rough sets are then proposed, and it is proved that properties of relation operations in the original information system and structural features of the system, such as approximations of arbitrary fuzzy sets and attribute reductions, are guaranteed in its image system under the condition of homomorphism.**

***Index Terms*—Attribute reduction, fuzzy relation mappings, fuzzy rough sets, homomorphism, information systems.**

I. Introduction

INFORMATION systems, which are also called knowledge representation systems, are formalisms to represent knowledge of some objects in terms of attributes and their values. Over the past few decades, many topics on information systems have been widely investigated, which include some successful applications in information processing, decision analysis, process control, and knowledge discovery [2]–[4], [6]–[20], [25]–[37], [39]–[45], [47]–[50]. Among these application topics, communication between information systems is an interesting and important one in the framework of granular computing [6], [28]. In light of the diversity of information systems, it is sometimes necessary to transmit information between information systems. For example, analog and digital conversion in signal processing, data fusion [16], [27], and equivalent attribute reductions and rule extractions [20], [38] need to handle information transformations. These problems motivate us to study the relationship between information systems.

Communication is directly related to the issue of transformations of information systems while preserving their basic properties. More formally, we propose the problem as follows: How to represent information transferred between different information systems. As explained in [6] and [28], communication allows one to translate the information contained in one granular world into the granularity of another granular world and thus provides a mechanism to exchange information with other granular worlds. From the mathematical viewpoint, this kind of communication can be explained as comparing some structures and properties of different information systems via mappings, which are useful tools to study the relationship between information systems. The notion of homomorphism based on rough sets [24], which was introduced by Grzymala-Busse in [9], was used to study information communication in [10]. In fact, a homomorphism can be viewed as a special mapping between information systems, and the notion of homomorphism on information systems is useful in aggregating sets of objects, attributes, and descriptors of the original system [10], [20], [28]. However, the study on the communication between information systems by the notion of homomorphism has not gained enough attention. Up to now, there are only a few research works that focus on this topic. In [10], Grzymala-Busse depicted some conditions which make some important attributes to be selective in terms of endomorphism of a complete information system. In [20], the features of superfluousness and reducts of complete information systems under some homomorphism is discussed. The above two pieces of work are both performed in the framework of Pawlak's rough sets [24], [25] and are mainly concentrated on the problem of attribute reduction under homomorphism. They did not discuss the issue of set approximations. We know that set approximations and attribute reductions in information systems are central notions in decision making, data analysis, reasoning about data, and other subfields of artificial intelligence [1]–[5], [12]–[19], [21]–[26], [30]–[37], [39]–[46]. For these reasons, Wang *et al*. further investigated some homomorphic properties of set approximations and attribute reductions in information systems based on binary relations and proved that attribute reductions in the original system and image system are equivalent to each other under the condition of homomorphism [38]. It is noted that their studies are only restricted to the generalized rough sets based on binary relations.

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In some situations, because of noises, inaccuracy, and human subjectivities, the boundaries of some attribute values are vague or ambiguous. For instance, low inflation rate, high pressure, and medium income are just a few illustrative cases. On the other hand, the values of some attributes in information systems could be real-valued or continuous. This kind of information system is quite different from the information system with discrete attribute values [38]. The method to study the communication between information systems with discrete attributes is not directly suitable to deal with continuous attributes. Numerical attributes can be segmented into several intervals with discretization algorithms [23] and then handled using the above methods, but the quality of corresponding results depends on discretization in this case. As argued in [46], discretization algorithms always lead to information loss, and this may influence the final decision results. In order to directly handle this type of information systems, a continuous attribute can be fuzzified by using the similarity degree of real values or by defining a mapping for such an attribute to induce a fuzzy binary relation [3]–[8], [11], [13]–[16], [26]–[32], [35], [39], [46]. How to investigate information communications between information systems with continuous attributes is the problem we would like to address in this paper.

The theory of fuzzy rough sets [5], as a generalization of classical rough sets [24], [25], has the capability of utilizing databases with vagueness and fuzziness by making use of the similarity degree of attribute values. This theory has been demonstrated to be useful in solving a variety of problems [4], [8], [11]–[15], [26], [27], [35], [39]. In this paper, we introduce fuzzy rough sets as a basic tool to study the communication between information systems and represent some new contributions to the development of this theory. By Zadeh's extension principle, we develop a method to define a fuzzy binary relation on a universe in terms of a fuzzy relation on another universe. In this sense, our method is a mechanism for communicating between two information systems. We then define the concepts of homomorphism between information systems based on fuzzy binary relations. Under the condition of homomorphism, some characteristics of relation operations in the original system and some structural features of the system, such as set approximations and attribute reductions, are guaranteed in its image system.

The remainder of this paper is organized as follows. In Section II, we review the relevant concepts in rough set theory. In Section III, we present the definitions of fuzzy relation mappings and investigate their properties. In Section IV, we introduce the concepts of homomorphism between information systems based on fuzzy binary relations and study their properties. Conclusions are drawn in Section V.

II. Preliminaries

This section mainly reviews some basic notions of rough and fuzzy rough sets related to this paper.

A. Rough Sets

An information system is a pair $IS = (U, C)$, where $U = \{x_1, x_2, \ldots, x_n\}$ is a nonempty finite set of objects and $C = \{a_1, a_2, \ldots, a_m\}$ is a nonempty finite set of attributes. With every subset of attributes $B \subseteq C$, we associate a binary relation $\mathrm{IND}(B)$, which is defined as

$$\mathrm{IND}(B) = \{(x, y) : a(x) \neq a(y), \forall a \in B\}.$$

$\mathrm{IND}(B)$ is obviously an equivalence relation and $\mathrm{IND}(B) = \cap_{a\in B} \mathrm{IND}(\{a\})$. By $[x]_B$, we denote the equivalence class of $\mathrm{IND}(B)$ including x. For any subset $X \subseteq U$, $\underline{B}X = \{x \in U : [x]_B \subseteq X\}$ and $\overline{B}X = \{x \in U : [x]_B \cap X \neq \oslash\}$ are called B-lower and upper approximations of X in IS, respectively.

An attribute $a \in B \subseteq C$ is superfluous in B if $\mathrm{IND}(B) = \mathrm{IND}(B - \{a\})$; otherwise, a is indispensable in B. The collection of all indispensable attributes in C is called the core of IS. We say that $B \subseteq C$ is independent in IS if every attribute in B is indispensable in B. $B \subseteq C$ is called a reduct in IS if B is independent and if $\mathrm{IND}(B) = \mathrm{IND}(C)$.

A decision system is a pair $\mathrm{DIS} = (U, C \cup \{d\})$, where d is the decision attribute, and C is a condition attribute set. The positive region of d relative to C is defined as

$$\mathrm{POS}_C(d) = \cup_{X\in U/d} \underline{B}X.$$

Let $a \in B \subseteq C$. If $\mathrm{POS}_B(d) = \mathrm{POS}_{B-\{a\}}(d)$, then a is called relatively dispensable in B; otherwise, a is said to be relatively indispensable in B. If every attribute in B is relatively indispensable in B, we say that $B \subseteq C$ is relatively independent in DIS. $B \subseteq C$ is called a relative reduct in DIS if B is relatively independent in DIS and $\mathrm{POS}_B(d) = \mathrm{POS}_C(d)$.

B. Fuzzy Sets and Fuzzy Logical Operators

Let $IS = (U, C = \{a_1, a_2, \ldots, a_m\})$ be an information system. If some attributes in C are real-valued or continuous, then each of them can induce a fuzzy relation by defining a similarity function. There are many discussions on how to construct fuzzy relations by real-valued or continuous attributes in the literature [8], [13]–[15], [27], [39]. In [8], a complete summarization on this topic is provided. For a detailed introduction to the notions, see these references.

Let U be a universal set. A fuzzy set A, or rather a fuzzy subset A of U, is defined by a function assigning to each element x of U a value $A(x) \in [0, 1]$. We denote by $F(U)$ the set of all fuzzy subsets of U. For any $A, B \in F(U)$, we say that A is contained in B, denoted by $A \subseteq B$, if $A(x) \leq B(x)$ for all $x \in U$, and we say that $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$. The support of a fuzzy set A is a set defined as $\mathrm{supp}(A) = \{x \in U | A(x) > 0\}$. Given $A, B \in F(U)$, the union of A and B, denoted as $A \cup B$, is defined by $(A \cup B)(x) = A(x) \vee B(x)$ for all $x \in U$; the intersection of A and B, which is denoted as $A \cap B$, is given by $(A \cap B)(x) = A(x) \wedge B(x)$ for all $x \in U$.

In [48], Zadeh proposed the extension principle, which has become an important tool in fuzzy set theory and its applications. Let U and V be two sets and f be a mapping from U to V; then,

f can be extended to a mapping from $F(U)$ to $F(V)$. That is

$$\tilde{f} : F(U) \to F(V), A| \to \tilde{f}(A) \in F(U) \quad \forall A \in F(U)$$

and $\tilde{f}(A)(y) = \sup_{x \in U, f(x)=y} \tilde{A}(x)$ for any $y \in V$.

Conversely, the mapping $f : U \to V$ can induce a mapping f^{-1} from $F(V)$ to $F(U)$ as follows:

$$\tilde{f}^{-1} : F(V) \to F(U), B \to \tilde{f}^{-1}(A) \in F(U) \quad \forall B \in F(V)$$

and $\tilde{f}^{-1}(B)(x) = B(f(x))$ for all $x \in U$.

A triangular norm, or shortly t-norm, is an increasing, associative, and commutative mapping $T : [0,1] \times [0,1] \to [0,1]$ that satisfies the boundary condition ($\forall x \in [0,1]$, $T(x,1) = x$).

The most popular continuous t-norms are the following:

1) the standard min operator $T_M(x,y) = \min\{x,y\}$;
2) the algebraic product $T_P(x,y) = x \cdot y$;
3) the bold intersection $T_L(x,y) = \max\{0,\ x + y - 1\}$.

A triangular conorm, or shortly t-conorm, is an increasing, associative, and commutative mapping $S : [0,1] \times [0,1] \to [0,1]$ that satisfies the boundary condition ($\forall x \in [0,1]$, $S(x,0) = x$).

Three well-known continuous t-conorms are given as follows:

1) the standard max operator $S_M(x,y) = \max\{x,y\}$;
2) the probabilistic sum $S_P(x,y) = x + y - x \cdot y$;
3) the bounded sum $S_L(x,y) = \min\{1,\ x + y\}$.

A negator N is a decreasing mapping $[0,1] \to [0,1]$ satisfying $N(0) = 1$ and $N(1) = 0$. The negator $N_s(x) = 1 - x$ is usually referred to as the standard negator. A negator N is called involutive iff $N(N(x)) = x$ for all $x \in [0,1]$, every involutive negator is continuous and strictly decreasing. Given a negator N, a t-norm T and a t-conorm S are dual with respect to N iff De Morgan laws are satisfied, i.e., $S(N(x), N(y)) = N(T(x,y)), T(N(x), N(y)) = N(S(x,y))$.

For every $A \in F(U)$, the symbol co_N will be used to denote fuzzy complement of A determined by a negator N, i.e., for every $x \in U, (co_N A)(x) = N(A(x))$.

Given a triangular norm T, for any $\alpha, \gamma \in [0,1]$, let $\vartheta(\alpha,\gamma) = \sup\{\theta \in [0,1] : T(\alpha,\theta) \le \gamma\}$; then, the binary operation ϑ on $[0,1]$ is called a R-implicator based on T. If T is lower semicontinuous, ϑ is called the residuation implication of T, or the T-residuated implication. The properties of T-residuated implication ϑ are listed in [46].

For a t-conorm S, the operator σ is defined as $\sigma(a,b) = \inf\{c \in [0,1] : S(a,c) \ge b\}$. If T and S are dual with respect to an involutive negator N, then ϑ and σ are dual with respect to the same involutive negator N, i.e., $\sigma(N(a), N(b)) = N(\vartheta(a,b))$ and $\vartheta(N(a), N(b)) = \sigma(N(a,b))$. The properties of σ are listed in [21] and [46].

C. Fuzzy Rough Sets

Let U be a nonempty universe. By a T-similarity relation R, we mean a fuzzy relation on U which is reflexive, symmetric, and T-transitive. The similarity class $[x]_R$ with $x \in U$ is a fuzzy set on U defined by $[x]_R(y) = R(x,y)$ for all $y \in U$; clearly, every similarity class is a normal fuzzy set.

Approximation operators of fuzzy sets can be summarized as the following four operators:

1) T-upper approximation operator:

$$\overline{R_T}A(x) = \sup_{u \in U} T(R(x,u), A(u))$$

2) S-lower approximation operator:

$$\underline{R_S}A(x) = \inf_{u \in U} S(N(R(x,u)), A(u))$$

3) σ-upper approximation operator:

$$\overline{R_\sigma}A(x) = \sup_{u \in U} \sigma(N(R(x,u)), A(u))$$

4) ϑ-lower approximation operator:

$$\underline{R_\vartheta}A(x) = \inf_{u \in U} \vartheta(R(x,u), A(u))$$

for every fuzzy set $A \in F(U)$, where R is an arbitrary fuzzy relation.

These approximation operators have been studied in detail in [46] from constructive, axiomatic, lattice, and fuzzy topological viewpoints. Clearly, the above lower and upper approximations are defined by membership functions; they are not explicitly defined as the union of some basic granular fuzzy sets, and the similarity classes cannot play this role since they are normal fuzzy sets. This motivates researchers to consider granulation of fuzzy points in [1].

Suppose U is a universe of discourse; T, S, and N are fuzzy logic operators defined in Section II-B, and T and S are dual with respect to N. Let R be a fuzzy relation on U. For $x \in U, \lambda \in (0,1], x_\lambda$ is a fuzzy set defined as $x_\lambda(y) = \begin{cases} \lambda, y = x \\ 0, y \ne x \end{cases}$, which is called a fuzzy point. We have $\overline{R_T}x_\lambda(y) = T(R(x,y),\lambda)$. According to the observation in [1], $\{\overline{R_T}x_\lambda : x \in U, \lambda \in (0,1]\}$ can be used as the basic information granules to reconstruct lower and upper approximation of fuzzy sets, That is, if R is a fuzzy T-similarity relation on U, then $\underline{R_\vartheta}A = \cup\{\overline{R_T}x_\lambda : \overline{R_T}x_\lambda \subseteq A\}$ and $\overline{R_T}A = \cup\{\overline{R_T}x_{A(x)} : x \in U\}$.

III. Fuzzy Relation Mappings and Their Properties

Mapping is a basic mathematical tool to communicate between two fuzzy sets. Similarly, by mapping, we could study the communication between information systems using fuzzy relations. In this section, we first define the notions of fuzzy relation mappings by Zadeh's extension principle and then study their properties. Let U and V be two universes. The classes of all fuzzy binary relations on U and V will be denoted by $\Re(U \times U)$ and $\Re(V \times V)$, respectively. Let us start with introducing the following concepts by Zadeh's extension principle.

Definition 3.1: Let $f : U \to V, u| \to f(u) \in V, u \in U$. By the extension principle, f can induce a mapping from $\Re(U \times U)$ to $\Re(V \times V)$ and a mapping from $\Re(V \times V)$ to $\Re(U \times U)$, i.e.,

$$\tilde{f} : \Re(U \times U) \to \Re(V \times V), R| \to \tilde{f}(R) \in \Re(V \times V), \forall R \in \Re(U \times U)$$

$$\tilde{f}(R)(x,y) = \begin{cases} \sup_{u\in f^{-1}(x)} \sup_{v\in f^{-1}(y)} R(u,v), & (x,y)\in f(U)\times f(U) \\ 0, & (x,y)\notin f(U)\times f(U). \end{cases}$$

$$\tilde{f}^{-1} : \Re(V\times V)\to \Re(U\times U),$$
$$P\to \tilde{f}^{-1}(P)\in \Re(U\times U), \forall P\in \Re(V\times V)$$
$$\tilde{f}^{-1}(P)(u,v) = P(f(u), f(v)).$$

Then, $\tilde{f}$ and $\tilde{f}^{-1}$ are called fuzzy relation mapping and inverse fuzzy relation mapping induced by f, respectively. $\tilde{f}(R)$ and $\tilde{f}^{-1}(P)$ are called fuzzy binary relations induced by f on V and U, respectively.

Remark 1: When R and P are crisp binary relations on U and V, respectively, the definitions of $\tilde{f}(R)$ and $\tilde{f}^{-1}(P)$ will be reduced to the definitions of the images of crisp binary relations in [38], respectively. For a given element $v\in U$, we know that $R(u,v), u\in U$ is a fuzzy set on U. If $f^{-1}(y)=v$, then $f(R)(x,y)=\sup_{x=f(u)} R(u,v)$ for any $x\in f(U)$, which coincides with Zadeh's extension principle. Thus, Definition 3.1 is an extension of the extension principle. So that there is no confusion in the subsequent discussion, we simply denote $\tilde{f}$ and $\tilde{f}^{-1}$ as f and f^{-1}, respectively.

Definition 3.2: Let U and V be two universes, $f: U\to V$ a mapping from U to V, and $R, R_1, R_2\in \Re(U\times U)$. Let $[x]_f = \{y\in U : f(y)=f(x)\}$; then, $\{[x]_f : x\in U\}$ is a partition on U. For any $x,y\in U$, if one of the following statements holds:

1) $R_1(u,v)\le R_2(u,v)$ for any $(u,v)\in [x]_f\times[y]_f$
2) $R_1(u,v)\ge R_2(u,v)$ for any $(u,v)\in [x]_f\times[y]_f$

then f is called consistent with respect to R_1 and R_2. If one of the following statements holds:

1) $R_1(u,v)< R_2(u,v)$ for any $(u,v)\in [x]_f\times[y]_f$
2) $R_1(u,v)> R_2(u,v)$ for any $(u,v)\in [x]_f\times[y]_f$

then f is strictly consistent with respect to R_1 and R_2. For any $u,s,\in [x]_f$ and $v,t\in[y]_f$, if $R(u,v)=R(u,t)$, then f is called predecessor-compatible with respect to R; if $R(u,v)=R(s,v)$, then f is called successor-compatible with respect to R; if $R(u,v)=R(s,t)$, then f is called compatible with respect to R.

It is easy to check that a function f is compatible with respect to R if and only if f is both successor-compatible and predecessor-compatible with respect to R. Let $A\in F(U)$ be a fuzzy set on U. Similar to Definition 3.2, we will say f is compatible with respect to A if $A(u)=A(v)$ for any $u,v\in[x]_f$, where $x\in U$. From Definition 3.2, an injection is trivially both a consistent and a compatible function.

Proposition 3.3: Let $R, R_1, R_2\in\Re(U\times U)$. If f is compatible with respect to R, R_1, and R_2, respectively. Then

1) f is compatible with respect to $R_1\cup R_2$;
2) f is compatible with respect to $R_1\cap R_2$;
3) f is compatible with respect to the complement of R.

Proof: Straightforward.

For a predecessor-compatible (successor-compatible) function, it has the similar properties for $R_1\cup R_2$, $R_1\cap R_2$ and the complement of R. Here, $\cup$ and $\cap$ represent max and min set operations, respectively.

Proposition 3.4: Let $R_1, R_2\in\Re(U\times U)$. If f is compatible with respect to R_1 and R_2 respectively, then f is consistent with respect to R_1 and R_2.

Proof: Straightforward.

Remark 2: If f is only predecessor-compatible, or successor-compatible with respect to R_1 and R_2, we cannot guarantee that f is consistent with respect to R_1 and R_2. It is also noted that the converse of Proposition 3.4 is not necessarily true.

The following theorem discusses properties of fuzzy binary relations under the relation mappings f and f^{-1}, respectively.

Theorem 3.5: Let $f: U\to V$, and f be surjective, $R\in\Re(U\times U)$, and $P\in\Re(V\times V)$. Then, we have the following.

1) If R is reflexive, then $f(R)$ is reflexive; if P is reflexive, then $f^{-1}(P)$ is reflexive.
2) If R is symmetric, then $f(R)$ is symmetric; If P is symmetric, then $f^{-1}(P)$ is symmetric.
3) If P is T-transitive, then $f^{-1}(P)$ is T-transitive.
4) If f is compatible with respect to R, then R is T-transitive if and only if $f(R)$ is T-transitive.

Proof: 1) Let R be reflexive; we need to prove that $f(R)$ is reflexive. Since f is surjective, it follows that for any $y\in V$, there must exist $x\in U$ such that $f(x)=y$. By the reflexivity of R, we have $R(x,x)=1$. From the definition of $f(R)$

$$f(R)(y,y) = \sup_{u\in f^{-1}(y)} \sup_{v\in f^{-1}(y)} R(u,v)\ge R(x,x)=1.$$

Thus, $f(R)$ is reflexive.

Let P be reflexive; we need to prove that $f^{-1}(P)$ is reflexive. For any $x\in U$, let $f(x)=y\in V$. By the reflexivity of P, we have $P(y,y)=1$. Thus

$$f^{-1}(P)(x,x) = P(f(x), f(x)) = P(y,y) = 1.$$

Hence, $f^{-1}(P)$ is reflexive.

2) Let R be symmetric; we need to prove that $f(R)$ is symmetric. For any $x,y\in V$

$$\begin{aligned} f(R)(x,y) &= \sup_{u\in f^{-1}(x)} \sup_{v\in f^{-1}(y)} R(u,v) \\ &= \sup_{v\in f^{-1}(y)} \sup_{u\in f^{-1}(x)} R(v,u) \\ &= f(R)(y,x) \end{aligned}$$

by the symmetry of R. Hence, $f(R)$ is symmetric.

Let P be symmetric; we need to prove that $f^{-1}(P)$ is symmetric. For any $u,v\in U$

$$\begin{aligned} f^{-1}(P)(u,v) &= P(f(u), f(v)) \\ &= P(f(v), f(u)) \\ &= f^{-1}(P)(v,u) \end{aligned}$$

by the symmetry of P. Hence, $f^{-1}(P)$ is symmetric.

3) Let P be T-transitive; for any $x,y,z\in U$

$$\begin{aligned} &T\left(f^{-1}(P)(x,y), f^{-1}(P)(y,z)\right) \\ &= T\left(P(f(x),f(y)), P(f(y),f(z))\right) \\ &\le P(f(x),f(z)) = f^{-1}(P)(x,z) \end{aligned}$$

by the transitivity of P. Hence, $f^{-1}(P)$ is transitive.

TABLE I
SIMILARITY RELATION R

T-1	x_1	x_2	x_3	x_4
x_1	1	0.25	0.25	0.5
x_2	0.25	1	1	0.25
x_3	0.25	1	1	0.25
x_4	0.5	0.25	0.25	1

4) $\Rightarrow$ For any $x, y, z \in V$, since f is surjective, it follows that there must exist $u_0, v_0, t_0 \in U$ such that

$$f(u_0) = x, f(v_0) = y, f(t_0) = z.$$

Since f is compatible with respect to R, we have

$$R(u_0, v_0) = R(u, v), R(v_0, t_0) = R(v, t)$$

for any $(u, v) \in f^{-1}(x) \times f^{-1}(y)$ and $(v, t) \in f^{-1}(y) \times f^{-1}(t)$. Hence

$$\begin{aligned} &T(f(R)(x, y), f(R)(y, z)) \\ &= T\Big(\Big(\sup_{u \in f^{-1}(x)} \sup_{v \in f^{-1}(y)} R(u, v)\Big), \\ &\Big(\sup_{s \in f^{-1}(y)} \sup_{t \in f^{-1}(z)} R(s, t)\Big)\Big) \\ &= T(R(u_0, v_0), R(v_0, t_0)) \leq R(u_0, t_0) \end{aligned}$$

by the transitivity of R. Similarly

$$f(R)(x, z) = \sup_{u \in f^{-1}(x)} \sup_{t \in f^{-1}(z)} R(u, t) = R(u_0, t_0).$$

Therefore, $T(f(R)(x, y), f(R)(y, z)) \leq f(R)(x, z)$, which implies $f(R)$ is T-transitive.

$\Leftarrow$ Since f is compatible with respect to R, we have $R(u_1, v_1) = R(u, v)$ for any $(u_1, v_1) \in f^{-1}(f(u)) \times f^{-1}(f(v))$.

Thus

$$f^{-1}(f(R))(u, v) = \sup_{f(u_1) = f(u)} \sup_{f(v_1) = f(v)} R(u_1, v_1) = R(u, v).$$

Therefore, $f^{-1}(f(R)) = R$. Since $f(R)$ is T-transitive, it follows from (3) and $f^{-1}(f(R)) = R$ that R is T-transitive. □

Remark 3: In general, a fuzzy relation mapping f can preserve the reflexivity and symmetry of a fuzzy relation, but it cannot preserve the transitivity of a fuzzy relation. This means that it needs more conditions to keep the transitivity of a fuzzy relation invariant. Below, we give an illustrative example. For simplification, we write Table 1 as T-1, Table 2 as T-2, and so on in data tables.

Example 3.1: Let $U = \{x_1, x_2, x_3, x_4\}, V = \{y_1, y_2, y_3\}$. R is a sup-min similarity relation on U given in Table I.

Define a mapping $f_1 : U \to V$ as follows:

$$\frac{x_1, x_2}{y_1} \quad \frac{x_3}{y_2} \quad \frac{x_4}{y_3}.$$

Then, $f_1(R)$ can easily be computed and is given in Table II.

TABLE II
RELATION f_1 (R)

T-2	y_1	y_2	y_3
y_1	1	1	0.5
y_2	1	1	0.25
y_3	0.5	0.25	1

TABLE III
RELATION f_2 (R)

T-3	y_1	y_2	y_3
y_1	1	0.25	0.5
y_2	0.25	1	0.25
y_3	0.5	0.25	1

Define a mapping $f_2 : U \to V$ as follows:

$$\frac{x_1}{y_1} \quad \frac{x_2, x_3}{y_2} \quad \frac{x_4}{y_3}.$$

Then, $f_2(R)$ can easily be computed and is given in Table III.

We can easily verify that f_1 is not compatible with respect to R and $f_1(R)$ is not sup-min transitive, while f_2 is compatible with respect to R and $f_2(R)$ is sup-min transitive. □

The following theorem discusses the properties of fuzzy relation operations under a fuzzy relation mapping f.

Theorem 3.6: Let $f : U \to V, R_1, R_2 \in \Re(U \times U)$; then, we have the following.

1) $f(R) = \oslash \Leftrightarrow R = \oslash$.
2) $f(R_1 \cup R_2) = f(R_1) \cup f(R_2)$.
3) $f(R_1 \cap R_2) \subseteq f(R_1) \cap f(R_2)$; if f is consistent with respect to R_1 and R_2, then the equality holds.
4) $R_1 \subseteq R_2 \Rightarrow f(R_1) \subseteq f(R_2)$.

Proof: 1) $\Rightarrow$ Since $f(R) = \oslash$, we have for any $x, y \in V$

$$\begin{aligned} &f(R)(x, y) = \sup_{u \in f^{-1}(x)} \sup_{v \in f^{-1}(y)} R(u, v) = 0 \\ &\Leftrightarrow \forall (u, v) \in f^{-1}(x) \times f^{-1}(y), R(u, v) = 0. \end{aligned}$$

Therefore, $R(u, v) = 0$ for any $u, v \in U$.

$\Leftarrow$ It is clearly true.

2)
$$\begin{aligned} f(R_1 \cup R_2)(x, y) &= \sup_{u \in f^{-1}(x)} \sup_{v \in f^{-1}(y)} (R_1 \cup R_2)(u, v) \\ &= \sup_{u \in f^{-1}(x)} \sup_{v \in f^{-1}(y)} (R_1(u, v) \vee R_2(u, v)) \\ &= (f(R_1) \cup f(R_2))(x, y). \end{aligned}$$

3) For any $x, y \in V$

$$\begin{aligned} f(R_1 \cap R_2)(x, y) &= \sup_{u \in f^{-1}(x)} \sup_{v \in f^{-1}(y)} (R_1 \cap R_2)(u, v) \\ &= \sup_{u \in f^{-1}(x)} \sup_{v \in f^{-1}(y)} \Big(R_1(u, v) \wedge R_2(u, v)\Big) \\ &\leq \Big(\sup_{u \in f^{-1}(x)} \sup_{v \in f^{-1}(y)} R_1(u, v)\Big) \\ &\wedge \Big(\sup_{u \in f^{-1}(x)} \sup_{v \in f^{-1}(y)} R_2(u, v)\Big) \\ &= (f(R_1) \cap f(R_2))(x, y). \end{aligned}$$

Now, we prove that if f is consistent with respect to R_1 and R_2, then the equality holds. By the above result, we only need to prove the inverse inclusion. Since f is consistent with respect to R_1 and R_2, it follows from Definition 3.2 that R_1 and R_2 satisfy one of the following conditions:

1) $R_1(u,v) \le R_2(u,v)$, and 2) $R_1(u,v) \ge R_2(u,v)$ for any $(u,v) \in f^{-1}(x) \times f^{-1}(y)$.

For case 1, we have

$$\begin{aligned} f(R_1 \cap R_2)(x,y) &= \sup_{u\in f^{-1}(x)} \sup_{v\in f^{-1}(y)} (R_1 \cap R_2)(u,v) \\ &= \sup_{u\in f^{-1}(x)} \sup_{v\in f^{-1}(y)} (R_1(u,v) \cap R_2(u,v)) \\ &= \sup_{u\in f^{-1}(x)} \sup_{v\in f^{-1}(y)} R_1(u,v) \\ &= f(R_1)(x,y) \end{aligned}$$

and

$$\begin{aligned} (f(R_1) \cap f(R_2))(x,y) &= f(R_1)(x,y) \wedge f(R_2)(x,y) \\ &= \Big(\sup_{u\in f^{-1}(x)} \sup_{v\in f^{-1}(y)} R_1(u,v)\Big) \\ &\quad \wedge \Big(\sup_{u\in f^{-1}(x)} \sup_{v\in f^{-1}(y)} R_2(u,v)\Big) \\ &= \sup_{u\in f^{-1}(x)} \sup_{v\in f^{-1}(y)} R_1(u,v) \\ &= f(R_1)(x,y). \end{aligned}$$

Hence, $f(R_1 \cap R_2) = f(R_1) \cap f(R_2)$. Similarly, for case 2, we also have $f(R_1 \cap R_2) = f(R_1) \cap f(R_2)$; therefore, we conclude the proof.

Proof: Straightforward. □

Remark 4: Equation (3) in Theorem 3.6 provides a sufficient condition to preserve the intersection operation of fuzzy relations under a fuzzy relation mapping f. In general, a fuzzy relation mapping f does not keep invariant the intersection of fuzzy relations. In the following we give an example to illustrate the case.

Example 3.2: Let $U = \{x_1, x_2, x_3, x_4\}$, and $V = \{y_1, y_2, y_3\}$. R_1 and R_2 are two fuzzy relations on U, which are given in Tables IV and V, respectively.

Define a mapping $f : U \to V$ as follows:

$$\frac{x_1, x_2 \qquad x_3, x_4}{y_1 \qquad\quad y_2}.$$

Then, $f(R_1)$, $f(R_2)$, and $f(R_1 \cap R_2)$ can be computed and are given in Tables VI, VI, and VIII, respectively. We can see that $f(R_1 \cap R_2) \subset f(R_1) \cap f(R_2)$. If we define a mapping $f : U \to V$ as

$$\frac{x_1, x_4 \qquad x_2 \qquad x_3}{y_1 \qquad\quad y_2 \qquad y_3}$$

we can verify that $f(R_1 \cap R_2) = f(R_1) \cap f(R_2)$.

TABLE IV
RELATION R_1

T-4	x_1	x_2	x_3	x_4
x_1	0.7	0.4	0.7	0.7
x_2	0.8	0.3	0.5	0.8
x_3	0.6	0.3	0.2	0.6
x_4	0.7	0.4	0.7	0.7

TABLE V
RELATION R_2

T-5	x_1	x_2	x_3	x_4
x_1	0.6	0.5	0.7	0.6
x_2	0.2	0.8	0.5	0.2
x_3	0.7	0.9	0.2	0.7
x_4	0.6	0.5	0.7	0.6

TABLE VI
RELATION $f(R_1)$

T-6	y_1	y_2
y_1	0.8	0.8
y_2	0.7	0.7

By Theorem 3.6, we can obtain the following corollary.

Corollary 3.7: Let $f : U \to V, R_1, R_2, \ldots, R_n \in \Re(U \times U)$; then

1) $f(\bigcup_{i=1}^n R_i) = \bigcup_{i=1}^n f(R_i)$.
2) $f(\bigcap_{i=1}^n R_i) \subseteq \bigcap_{i=1}^n f(R_i)$; if f is consistent with respect to any two of fuzzy relations $R_1, R_2, \ldots, R_n$, then the equality holds.

Corollary 3.8: If f is compatible with respect to each of fuzzy relations $R_1, R_2, \ldots, R_n \in \Re(U \times U)$, then $f(\bigcap_{i=1}^n R_i) = \bigcap_{i=1}^n f(R_i)$.

The following theorem discusses operations of fuzzy relations under an inverse fuzzy relation mapping f^{-1}.

Theorem 3.9: Let $f : U \to V, P, P_1, P_2 \in \Re(V \times V)$; then

1) $f^{-1}(\oslash) = \oslash$.
2) If f is surjective, $f^{-1}(P) = \oslash \Leftrightarrow P = \oslash$.
3) $f^{-1}(P_1 \cup P_2) = f^{-1}(P_1) \cup f^{-1}(P_2)$.
4) $f^{-1}(P_1 \cap P_2) = f^{-1}(P_1) \cap f^{-1}(P_2)$.
5) $P_1 \subseteq P_2 \Rightarrow f^{-1}(P_1) \subseteq f^{-1}(P_2)$.

Proof: Straightforward. □

Theorem 3.10: Let $f : U \to V, R \in \Re(U \times U), P \in \Re(V \times V)$; then

1) $f(f^{-1}(P)) \subseteq P$; if f is surjective, then the equality holds.
2) $f^{-1}(f(R)) \supseteq R$; the equality holds if and only if f is compatible with respect to R.

Proof: 1) For $(x,y) \in f(U) \times f(U) \subseteq V \times V, f^{-1}(x) \neq \oslash$ and $f^{-1}(y) \neq \oslash$. Thus

$$\begin{aligned} f(f^{-1}(P))(x,y) &= \sup_{u\in f^{-1}(x)} \sup_{v\in f^{-1}(y)} f^{-1}(P)(u,v) \\ &= \sup_{u\in f^{-1}(x)} \sup_{v\in f^{-1}(y)} P(f(u), f(v)) \\ &= P(x,y). \end{aligned}$$

For $(x, y) \notin f(U) \times f(U)$ satisfying $(x, y) \in V \times V$, $f\left(f^{-1}(P)\right)(x, y) = 0$ by the Definition 3.1. Hence $f\left(f^{-1}(P)\right)(x, y) \subseteq P(x, y)$.

2) Since

$$\begin{aligned} f^{-1}(f(R))(u, v) &= f(R)(f(u), f(v)) \\ &= \sup_{f(x)=f(u)} \sup_{f(y)=f(v)} R(x, y) \geq R(u, v) \end{aligned}$$

for any $(u, v) \in U \times U$, we have $f^{-1}(f(R)) \supseteq R$.

If f is compatible with respect to R, we have $R(x, y) = R(u, v)$ for any $(x, y) \in f^{-1}(f(u)) \times f^{-1}(f(v))$.

Thus

$$f^{-1}(f(R))(u, v) = \sup_{f(x)=f(u)} \sup_{f(y)=f(v)} R(x, y) = R(u, v).$$

Therefore, $f^{-1}(f(R)) = R$, and the sufficiency holds.

Suppose $f^{-1}(f(R)) = R$. If there are two pairs of elements $(x, y), (u, v) \in U \times U$ satisfying $f(x) = f(u)$ and $f(y) = f(v)$, but $R(x, y) \neq R(u, v)$. Without loss of generality, we assume that $R(x, y) < R(u, v)$. On the other hand, by $f^{-1}(f(R)) = R$, we have

$$\begin{aligned} R(x, y) &= f^{-1}(f(R))(x, y) \\ &= f(R)(f(x), f(y)) \\ &= \sup_{f(u)=f(x)} \sup_{f(u)=f(x)} R(u, v) \geq R(u, v). \end{aligned}$$

This is a contradiction to $R(x, y) < R(u, v)$, and we get the necessity. □

By Proposition 3.3 and Theorem 3.10, we can obtain the following corollary.

Corollary 3.11: Let $f : U \to V, R_1, R_2, \ldots, R_n \in \Re(U \times U)$, and $P_1, P_2, \ldots, P_n \in \Re(V \times V)$; then

1) $f\left(f^{-1}\left(\bigcap_{i=1}^n P_i\right)\right) \subseteq \bigcap_{i=1}^n P_i$; if f is surjective, then the equality holds;
2) $f^{-1}\left(f\left(\bigcap_{i=1}^n R_i\right)\right) \supseteq \bigcap_{i=1}^n R_i$; the equality holds if f is compatible with respect to R_i $(i \leq n)$.

The following theorem shows that under a certain condition, the images of fuzzy information granules on U and V can be viewed as the fuzzy information granules on V and U, respectively.

Theorem 3.12: Let f be a mapping from U to $V, x \in U$ and $\lambda \in [0, 1]$. Let $\overline{R_T} x_\lambda$ be a T-fuzzy information granule on U. Then

1) $f\left(\overline{R_T} x_\lambda\right) \subseteq \overline{f(R)_T} f(x)_\lambda$; if f is successor-compatible with respect to R, then the equality holds. In particular, if T is a strictly increasing function, then the equality holds if and only if f is successor-compatible with respect to R.
2) $f^{-1}\left(f\left(\overline{R_T} x_\lambda\right)\right) \supseteq \overline{R_T} x_\lambda$; If f is predecessor-compatible with respect to R, then the equality holds. In particular, if T is a strictly increasing function, then the equality holds if and only if f is predecessor-compatible with respect to R.

Proof: 1) By the definition of $\overline{R_T} x_\lambda$, it follows from the properties of T that

$$\begin{aligned} f\left(\overline{R_T} x_\lambda\right)(y) &= \sup_{v \in f^{-1}(y)} \overline{R_T} x_\lambda(v) \\ &= \sup_{v \in f^{-1}(y)} T(R(x, v), \lambda) \\ &= T\Big(\sup_{v \in f^{-1}(y)} R(x, v), \lambda\Big) \end{aligned}$$

and

$$\begin{aligned} \overline{f(R)_T} f(x)_\lambda(y) &= T(f(R)(f(x), y), \lambda) \\ &= T\Big(\sup_{u \in f^{-1}(f(x))} \sup_{v \in f^{-1}(y)} R(u, v), \lambda\Big). \end{aligned}$$

Hence, $f\left(\overline{R_T} x_\lambda\right)(y) \subseteq \overline{f(R)_T} f(x)_\lambda(y)$.

If f is successor-compatible with respect to R, then we have $\sup_{u \in f^{-1}(f(x))} \sup_{v \in f^{-1}(y)} R(u, v) = \sup_{v \in f^{-1}(y)} R(x, v)$. Thus, $f\left(\overline{R_T} x_\lambda\right) = \overline{f(R)_T} f(x)_\lambda$.

On the contrary, let T be a strictly increasing function and $f\left(\overline{R_T} x_\lambda\right) = \overline{f(R)_T} f(x)_\lambda$. As shown above, we have

$$\begin{aligned} f\left(\overline{R_T} x_\lambda\right) &= T\Big(\sup_{v \in f^{-1}(y)} R(x, v), \lambda\Big) \\ &= T\Big(\sup_{u \in f^{-1}(f(x))} \sup_{v \in f^{-1}(y)} R(u, v), \lambda\Big) \\ &= \overline{f(R)_T} f(x)_\lambda. \end{aligned}$$

Since T is a strictly increasing function, it follows that $\sup_{v \in f^{-1}(y)} R(x, v) = \sup_{u \in f^{-1}(f(x))} \sup_{v \in f^{-1}(y)} R(u, v)$. Thus, $R(x, v) = \sup_{u \in f^{-1}(f(x))} R(u, v)$. By the arbitrariness of $x \in U$, we know that f is successor-compatible with respect to R.

2)

$$\begin{aligned} f^{-1}\left(f\left(\overline{R_T} x_\lambda\right)\right)(y) &= f\left(\overline{R_T} x_\lambda\right)(f(y)) \\ &= \sup_{u \in f^{-1}(f(y))} \overline{R_T} x_\lambda(u) \\ &= \sup_{u \in f^{-1}(f(y))} T(R(x, u), \lambda) \\ &= T\Big(\sup_{u \in f^{-1}(f(y))} R(x, u), \lambda\Big) \\ &\supseteq T(R(x, y), \lambda) \\ &= \overline{R_T} x_\lambda(y). \end{aligned}$$

If f is predecessor-compatible with respect to R, then $\sup_{u\in f^{-1}(f(y))} R(x,u) = R(x,y)$. Thus, the equality holds.

On the contrary, let T be a strictly increasing function, and $f^{-1}\left(f\left(\overline{R_T}x_\lambda\right)\right) = \overline{R_T}x_\lambda$. As shown above, we have

$$\begin{aligned} f^{-1}\left(f\left(\overline{R_T}x_\lambda\right)\right)(y) &= T\Big(\sup_{u\in f^{-1}(f(y))} R(x,u), \lambda\Big) \\ &= T(R(x,y),\lambda) = \overline{R_T}x_\lambda(y). \end{aligned}$$

Since T is a strictly increasing function, it follows that $\sup_{u\in f^{-1}(f(y))} R(x,u) = R(x,y)$. By the arbitrariness of $y \in U$, we know that f is predecessor-compatible with respect to R. □

The following theorem discusses the issue of approximations under fuzzy relation mappings f and f^{-1}, respectively.

Theorem 3.13: Let $f: U \to V, R \in \Re(U\times U), P \in \Re(V\times V), A \in F(U)$, and $B \in F(V)$; then, we have the following.

1) If f is compatible with respect to R, then $f\left(\underline{apr}_R A\right) \subseteq \underline{apr}_{f(R)} f(A)$; in particular, if f is also compatible with respect to A, then the equality holds.
2) If f is compatible with respect to R, then $f\left(\overline{apr}_R A\right) \subseteq \overline{apr}_{f(R)} f(A)$; in particular, if f is also compatible with respect to A, then the equality holds.
3) For $\xi \in [0,1]$, let $\Delta_\xi(u) = \xi, \forall u \in U$. Then $f\left(\underline{apr}_R \Delta_\xi\right) = \underline{apr}_{f(R)} f(\Delta_\xi) = \Delta_\xi$.
4) For $\xi \in [0,1]$, let $\Delta_\xi(u) = \xi, \forall u \in U$. Then $f\left(\overline{apr}_R \Delta_\xi\right) = \overline{apr}_{f(R)} f(\Delta_\xi) = \Delta_\xi$.
5) $f^{-1}\left(\underline{apr}_P B\right) \subseteq \underline{apr}_{f^{-1}(P)} f^{-1}(B)$; the equality holds iff f is surjective.
6) $f^{-1}\left(\overline{apr}_P B\right) \supseteq \overline{apr}_{f^{-1}(P)} f^{-1}(B)$; the equality holds iff f is surjective.

Here $\underline{apr}_R$ is referred to as lower approximation operator such as $\underline{R_S}$ or $\underline{R_\vartheta}$. $\overline{apr}_R$ is referred to as upper approximation operator such as $\overline{R_T}$ or $\overline{R_\sigma}$.

Proof: 1) For any $x, s \in V$, since f is compatible with respect to R, it follows from Definition 3.2 that $R(y,z) = R(y_0,z_0)$ for any $(y,z),(y_0,z_0) \in f^{-1}(x)\times f^{-1}(s)$. For S-lower approximation operator, we have

$$\begin{aligned} f\left(\underline{apr}_R A\right)(x) &= \sup_{y\in f^{-1}(x)} \left(\underline{apr}_R A\right)(y) \\ &= \sup_{y\in f^{-1}(x)} \left\{\inf_{z\in U} S(N(R(y,z)),A(z))\right\} \\ &= \inf_{z\in U} S(N(R(y_0,z)),A(z)) \\ &= \left(\underline{apr}_R A\right)(y_0), y_0 \in f^{-1}(x) \end{aligned}$$

and

$$\begin{aligned} \left(\underline{apr}_{f(R)} f(A)\right)(x) &= \inf_{s\in V} S(N(f(R)(x,s)), f(A)(s)) \\ &= \inf_{s\in V} S\Big(N\Big(\sup_{y\in f^{-1}(x)}\sup_{z\in f^{-1}(s)} R(y,z)\Big), \sup_{w\in f^{-1}(s)} A(w)\Big) \\ &= \inf_{s\in V} S\Big(N(R(y_0,z_0)), \Big(\sup_{w\in f^{-1}(s)} A(w)\Big)\Big) \\ &\geq \inf_{s\in V} S(N(R(y_0,z_0)),A(z_0))\left(z_0 \in f^{-1}(s)\right) \\ &= \inf_{z_0\in U} S(N(R(y_0,z_0)),A(z_0)) \\ &= \left(\underline{apr}_R A\right)(y_0), y_0 \in f^{-1}(x). \end{aligned}$$

Thus, $f\left(\underline{apr}_R A\right) \subseteq \underline{apr}_{f(R)} f(A)$. Specially, if f is also compatible with respect to A, we have $A(w) = A(z)$ for any $w, z \in f^{-1}(s)$, which implies $\sup_{w\in f^{-1}(s)} A(w) = A(z)$. Similarly, we can conclude that $f\left(\underline{apr}_R A\right) = \underline{apr}_{f(R)} f(A)$.

Similar to the above proof, we also have the same result for $\vartheta-$lower approximation operator.

2) For $T-$upper approximation operator, we have

$$\begin{aligned} f\left(\overline{apr}_R A\right)(x) &= \sup_{y\in f^{-1}(x)} \left(\overline{apr}_R A\right)(y) \\ &= \sup_{y\in f^{-1}(x)} \left\{\sup_{z\in U} T(R(y,z),A(z))\right\} \\ &= \sup_{z\in U} \left\{\sup_{y\in f^{-1}(x)} T(R(y,z),A(z))\right\} \\ &= \sup_{z\in U} T(R(y_0,z),A(z)) \\ &= \left(\overline{apr}_R A\right)(y_0), y_0 \in f^{-1}(x) \end{aligned}$$

and

$$\begin{aligned} \left(\overline{apr}_{f(R)} f(A)\right)(x) &= \sup_{s\in V} T(f(R)(x,s), f(A)(s)) \\ &= \sup_{s\in V} T\Big(\sup_{y\in f^{-1}(x)}\sup_{z\in f^{-1}(s)} R(y,z), \sup_{w\in f^{-1}(s)} A(w)\Big) \\ &= \sup_{s\in V} T\Big((R(y_0,z_0)), \sup_{w\in f^{-1}(s)} A(w)\Big) \\ &\geq \sup_{s\in V} T(R(y_0,z_0),A(z_0))\left(z_0 \in f^{-1}(s)\right) \\ &= \sup_{z_0\in U} T(R(y_0,z_0),A(z_0)) \\ &= \left(\overline{apr}_R A\right)(y_0), y_0 \in f^{-1}(x). \end{aligned}$$

Thus, $f\left(\overline{apr}_R A\right) \subseteq \overline{apr}_{f(R)} f(A)$. Similar to the above proof, we have the same result for σ-upper approximation operator.

For S-lower approximation operator, we have

$$\begin{aligned} \underline{apr}_R \Delta_\xi(x) &= \inf_{y\in U} S(N(R(x,y)),\Delta_\xi(y)) \geq \inf_{y\in U} \Delta_\xi(y) \\ &= \xi = \Delta_\xi(x) \end{aligned}$$

which implies $\Delta_\xi \subseteq \underline{apr}_R \Delta_\xi$. Hence, $\Delta_\xi = \underline{apr}_R \Delta_\xi$. Thus

$$f\left(\underline{apr}_R \Delta_\xi\right)(x) = f(\Delta_\xi)(x) = \sup_{y\in f^{-1}(x)} \Delta_\xi(y) = \xi.$$

Since

$$\underline{apr}_{f(R)} f(\Delta_\xi)(x) = \inf_{y\in U} S(N(f(R)(x,y)), f(\Delta_\xi)(y))$$
$$= \inf_{y\in U} S\Big(N\big(\sup_{u\in f^{-1}(x)} \sup_{v\in f^{-1}(y)} R(u,v)\big), f(\Delta_\xi)(y)\Big)$$
$$\geq f(\Delta_\xi)(y) = \xi = f(\Delta_\xi)(x)$$

we have $\underline{apr}_{f(R)} f(\Delta_\xi) = f(\Delta_\xi)$, which implies

$$\underline{apr}_{f(R)} f(\Delta_\xi) = f(\Delta_\xi) = \xi.$$

Similar to the above proof, we have the same result for $\vartheta-$lower approximation operator. Thus, we complete the proof.

4) Similar to the proof of (3).

5) For S-lower approximation operator, we have

$$f^{-1}\big(\underline{apr}_P B\big)(x) = \underline{apr}_P B(f(x))$$
$$= \inf_{y\in V} S(N(P(f(x),y)), B(y))$$

and

$$\underline{apr}_{f^{-1}(P)} f^{-1}(B)(x)$$
$$= \inf_{z\in U} S\left(N\left(f^{-1}(P)(x,z)\right), f^{-1}(B)(z)\right)$$
$$= \inf_{z\in U} S(N(P(f(x), f(z))), B(f(z)))$$
$$\geq \inf_{y\in V} S(N(P(f(x),y)), B(y))$$

by $f(U) \subseteq V$. Thus, $f^{-1}\big(\underline{apr}_P Y\big) \subseteq \underline{apr}_{f^{-1}(P)} f^{-1}(Y)$. If f is surjective, $f(U) = V$, which implies $f^{-1}\big(\underline{apr}_P B\big) = \underline{apr}_{f^{-1}(P)} f^{-1}(B)$.

On the contrary, assume that f is not surjective; then $f(U) \subseteq V$. According to the above proof, we have $f^{-1}\left(\underline{apr}_P B\right) \subseteq \underline{apr}_{f^{-1}(P)} f^{-1}(B)$, which is a contradiction. Similar to the above proof, we have the same result for ϑ-lower approximation operator. We complete the proof.

6) Similar to the proof of (5). □

IV. Homomorphism Between Information Systems With Fuzzy Relations

In this section, based on the results of the previous sections, we introduce the notions of homomorphism to study the communication between information systems with fuzzy relations, and investigate some properties of this type of information systems under the condition of homomorphism. However, the idea of homomorphism in this section has no closed link with the ones in standard algebra such as group theory or ring theory. Homomorphism employs the same idea to set up relationship between two sets. We first introduce the notions of fuzzy relation information systems.

Definition 4.1: Let U and V be finite universes, $f: U \to V$ a mapping from $U$to V, and $\mathbf{R} = \{R_1, R_2, \ldots, R_n\}$ a family of fuzzy binary relations on U; let $f(\mathbf{R}) = \{f(R_1), f(R_2), \ldots, f(R_n)\}$. Then, the pair $(U, \mathbf{R})$ is referred to as a fuzzy relation information system, and the pair $(V, f(\mathbf{R}))$ is referred to as a $f-$induced fuzzy relation information system of $(U, \mathbf{R})$.

Definition 4.2: Let U and V be finite universes, $f: U \to V$ a mapping from U to V, and $\mathbf{P} = \{P_1, P_2, \ldots, P_n\}$ a family of fuzzy binary relations on V. Let $f^{-1}(\mathbf{P}) = \left\{f^{-1}(P_1), f^{-1}(P_2), \ldots, f^{-1}(P_n)\right\}$. Then, the pair $(V, \mathbf{P})$ is referred to as a fuzzy relation information system, and the pair $\left(U, f^{-1}(\mathbf{P})\right)$ is referred to as a f-induced fuzzy relation information system of $(V, \mathbf{P})$.

Definition 4.3: Let $(U, \mathbf{R})$ be a fuzzy relation information system and $(V, f(\mathbf{R}))$ a f-induced fuzzy relation information system of $(U, \mathbf{R}), R_i \in \mathbf{R}, i \leq n$. f is referred to as a homomorphism from $(U, \mathbf{R})$ to $(V, f(\mathbf{R}))$ if f satisfies the following conditions:

1) $f\left(\bigcap_{i=1}^n R_i\right) = \bigcap_{i=1}^n f(R_i)$.
2) $f\left(\bigcap_{i=1}^n R_i\right) = \bigcap_{i=1}^n f(R_i)$.

Definition 4.4: Let $\left(U, f^{-1}(\mathbf{P})\right)$ be a f-induced fuzzy relation information system of $(V, \mathbf{P})$. If f is surjective, then f^{-1} is referred to as a homomorphism from $(V, \mathbf{P})$ to $\left(U, f^{-1}(\mathbf{P})\right)$.

Theorem 4.5: Let $(U, \mathbf{R})$ be a fuzzy relation information system and $(V, f(\mathbf{R}))$ a f-induced fuzzy relation information system of $(U, \mathbf{R})$. If $\forall R_i, R_j \in \mathbf{R}$, f is consistent with respect to R_i and R_j, then f is a homomorphism from $(U, \mathbf{R})$ to $(V, f(\mathbf{R}))$.

Proof: It follows immediately from Corollary 3.7 and Definition 4.3.

By Proposition 3.4 and Theorem 4.5, we get the following corollary.

Corollary 4.6: Let $(U, \mathbf{R})$ be a fuzzy relation information system and $(V, f(\mathbf{R}))$ a f-induced fuzzy relation information system of $(U, \mathbf{R})$. If $\forall R_i \in \mathbf{R}$, $f$is compatible with respect to R_i, then f is a homomorphism from $(U, \mathbf{R})$ to $(V, f(\mathbf{R}))$.

In order to identify homomorphism satisfying different conditions, we make the following appointments.

1) If $\forall R_i, R_j \in \mathbf{R}$, f is consistent (respectively, strictly consistent) with respect to R_i and R_j, we call f a consistent (respectively, strictly consistent) homomorphism.
2) A homomorphism satisfying the condition that f is compatible with respect to each fuzzy relation $R_i \in \mathbf{R}$ is called compatible.

Remark 5: After introducing the notions of homomorphism, all the theorems and corollaries in which the equality "=" holds in the above sections may be viewed as the properties of homomorphism.

The following theorems and corollaries reveal the nature of a homomorphism from $(U, \mathbf{R})$ to $(V, f(\mathbf{R}))$.

Theorem 4.7: Let $(U, \mathbf{R})$ be a fuzzy relation information system, $(V, f(\mathbf{R}))$ a f-induced fuzzy relation information system of $(U, \mathbf{R})$, and f a compatible homomorphism from $(U, \mathbf{R})$ to $(V, f(\mathbf{R}))$. Then, $\forall R_i \in \mathbf{R}$, where

1) R_i is reflexive if and only if $f(R_i)$ is reflexive;
2) R_i is symmetric if and only if $f(R_i)$ is symmetric;
3) R_i is transitive if and only if $f(R_i)$ is transitive;
4) the numbers of binary relations in $\mathbf{R}$ and $f(\mathbf{R})$ are equal to each other, that is, f is one-to-one with respect to R_i;

5) if f is surjective (not bijective), then the number of objects in V is less than in U, i.e., f is not one-to-one with respect to $x \in U$.

Proof: Since f is a compatible homomorphism from $(U, \mathbf{R})$ to $(V, f(\mathbf{R}))$, we know that $\forall R_i \in \mathbf{R}$, f is compatible with respect to R_i.

(1) If R_i is reflexive, by Theorem 3.5, $f(R_i)$ is also reflexive. Letting R_i be not reflexive, and assuming that $f(R_i)$ is reflexive, by Theorem 3.5 and Theorem 3.10 (2), we know that R_i is reflexive. This is a contradiction. Therefore, $f(R_i)$ is also not reflexive.

(2) and (3) are similar to the proof of (1).

(4) Let $R_1 \neq R_2$. Assume that $f(R_1) = f(R_2)$. Then, $f^{-1}(f(R_1)) = f^{-1}(f(R_2))$. According to Theorem 3.10 (2), $f^{-1}(f(R_1)) = R_1$ and $f^{-1}(f(R_2)) = R_2$. It follows that $R_1 = R_2$. This is a contradiction.

(5) Straightforward. □

Remark 6: From (1)–(3) of this theorem, we can see that the reflexivity, symmetry, and transitivity of a fuzzy relation are kept in the communication between two information systems under a compatible homomorphism f. By (4) and (5) of this theorem, we know that a compatible homomorphism f on information systems is a useful tool to aggregate sets of objects. It should be pointed out that a strictly consistent homomorphism also has the same properties as (4) and (5) listed above.

Remark 7: Parallel to Theorem 4.7, there are similar properties for a homomorphism from $(V, \mathbf{P})$ to $(U, f^{-1}(\mathbf{P}))$.

Definition 4.8: Let $(U, \mathbf{R})$ be a fuzzy relation information system, $R \subseteq \mathbf{R}$ and $\mathbf{S} \subseteq \mathbf{R}$. R is said to be superfluous in R if $\cap\mathbf{R} = \cap(\mathbf{R} - \{R\})$; otherwise, it is indispensable. The subset $\mathbf{S}$ is referred to as a reduct of $\mathbf{R}$ if $\mathbf{S}$ satisfies the following conditions:

1) $\cap\mathbf{S} = \cap\mathbf{R}$;
2) $\forall R_i \in \mathbf{S}, \cap\mathbf{S} \subset \cap(\mathbf{S} - \{R_i\})$.

Let $\mathrm{Re}\, d(\mathbf{R})$ be the collection of all reducts of $\mathbf{R}$, and let $Core(\mathbf{R}) = \cap \mathrm{Re}\, d(\mathbf{R})$, similar to the counterpart in Pawlak's rough sets, $Core(\mathbf{R})$ is the collection of all indispensable elements in $\mathbf{R}$ and is called the core of $\mathbf{R}$.

Theorems 4.9 and 4.10 show that under the condition of a strictly consistent homomorphism, or a compatible homomorphism f, the image of reduct of the original system is also a reduct of image system.

Theorem 4.9: Let $(U, \mathbf{R})$ be a fuzzy relation information system, $(V, f(\mathbf{R}))$ a f-induced fuzzy relation information system of $(U, \mathbf{R})$, f a strictly consistent homomorphism from $(U, \mathbf{R})$ to $(V, f(\mathbf{R}))$ and $\mathbf{S} \subseteq \mathbf{R}$. Then, $\mathbf{S}$ is a reduct of $\mathbf{R}$ if and only if $f(\mathbf{S})$ is a reduct of $f(\mathbf{R})$.

Proof: $\Rightarrow$ Since $\mathbf{S}$ is a reduct of $\mathbf{R}$, we have $\cap\mathbf{S} = \cap\mathbf{R}$. Hence $f(\cap\mathbf{S}) = f(\cap\mathbf{R})$. Since f is a consistent homomorphism from $(U, \mathbf{R})$ to $(V, f(\mathbf{R}))$, by Definition 4.3 and Corollary 3.7 (2), we have $\cap f(\mathbf{S}) = \cap f(\mathbf{R})$. Assume that $\exists R_i \in \mathbf{S}$ such that $\cap(f(\mathbf{S}) - f(R_i)) = \cap f(\mathbf{S})$. Because of $f(\mathbf{S}) - f(R_i) = f(\mathbf{S} - \{R_i\})$, we have $\cap(f(\mathbf{S}) - f(R_i)) = \cap f(\mathbf{S} - \{R_i\}) = \cap f(\mathbf{S}) = \cap f(\mathbf{R})$. Similarly, by Definition 4.3 and Corollary 4.7(2), it follows that $f(\cap(\mathbf{S} - R_i)) = f(\cap\mathbf{R})$. Again, since $\mathbf{S}$ is a reduct of $\mathbf{R}$, we have $\cap\mathbf{S} \neq \cap(\mathbf{S} - \{R_i\})$. Thus, there must be $x_0, y_0 \in U$ such that $\cap\mathbf{S}(x_0, y_0) < \cap(\mathbf{S} - \{R_i\})(x_0, y_0)$, which implies

$$\begin{aligned} & f(\cap(\mathbf{S} - \{R_i\}))(f(x_0), f(y_0)) \\ & = \sup_{u \in f^{-1}(f(x_0))} \sup_{v \in f^{-1}(f(y_0))} \cap(\mathbf{S} - \{R_i\})(u, v) \\ & > \sup_{u \in f^{-1}(f(x_0))} \sup_{v \in f^{-1}(f(y_0))} \cap\mathbf{S}(u, v) \\ & = f(\cap\mathbf{S})(f(x_0), f(y_0)) \\ & = f(\cap\mathbf{R})(f(x_0), f(y_0)) \end{aligned}$$

by the fact that f is a strictly consistent homomorphism. This is a contradiction.

$\Leftarrow$ Let $f(\mathbf{S})$ be a reduct of $f(\mathbf{R})$; then, $\cap f(\mathbf{S}) = \cap f(\mathbf{R})$. Since f is a consistent homomorphism from $(U, \mathbf{R})$ to $(V, f(\mathbf{R}))$, by Definition 4.3 and Corollary 3.7(2), we have $f(\cap\mathbf{S}) = f(\cap\mathbf{R})$. Assume that $\cap\mathbf{S} \supseteq \cap\mathbf{R}$, there must exist $x_0, y_0 \in U$ such that $\cap\mathbf{R}(x_0, y_0) < \cap\mathbf{S}(x_0, y_0)$. By the strictly consistence of homomorphism, we have

$$\begin{aligned} f(\cap\mathbf{S})(f(x_0), f(y_0)) & = \sup_{u \in f^{-1}(f(x_0))} \sup_{v \in f^{-1}(f(y_0))} \cap\mathbf{S}(u, v) \\ & > \sup_{u \in f^{-1}(f(x_0))} \sup_{v \in f^{-1}(f(y_0))} \cap\mathbf{R}(u, v) \\ & = f(\cap\mathbf{R})(f(x_0), f(y_0)). \end{aligned}$$

This is a contradiction. Thus, $\cap\mathbf{S} = \cap\mathbf{R}$. Assume that $\exists R_i \in \mathbf{S}$ such that $\cap(\mathbf{S} - \{R_i\}) = \cap\mathbf{R}$; then, $f(\cap(\mathbf{S} - R_i)) = f(\cap\mathbf{R})$. Again, by Definition 4.3 and Corollary 3.7(2), we have $\cap f(\mathbf{S} - \{R_i\}) = \cap f(\mathbf{R})$. Hence, $\cap(f(\mathbf{S}) - f(R_i)) = \cap f(\mathbf{R})$. This is a contradiction to the fact that $f(\mathbf{S})$ is a reduct of $f(\mathbf{R})$. This completes the proof of this theorem. □

Theorem 4.10: Let $(U, \mathbf{R})$ be a fuzzy relation information system, $(V, f(\mathbf{R}))$ a f-induced fuzzy relation information system of $(U, \mathbf{R})$, f a compatible homomorphism from $(U, \mathbf{R})$ to $(V, f(\mathbf{R}))$ and $\mathbf{S} \subseteq \mathbf{R}$. Then, $\mathbf{S}$ is a reduct of $\mathbf{R}$ if and only if $f(\mathbf{S})$ is a reduct of $f(\mathbf{R})$.

Proof: It follows immediately from Definitions 4.3 and Corollaries 3.8, 3.11(2).

By Theorems 4.9 and 4.10, we immediately get the following corollary.

Corollary 4.11: Let $(V, f(\mathbf{R}))$ be a f-induced fuzzy relation information system of $(U, \mathbf{R})$, $R \in \mathbf{R}$, and $\mathbf{S} \subseteq \mathbf{R}$. If f is a strictly consistent homomorphism, or a compatible homomorphism from $(U, \mathbf{R})$ to $(V, f(\mathbf{R}))$, then we have following.

1) R is indispensable in $\mathbf{R}$ if and only if $f(R)$ is indispensable in $f(\mathbf{R})$.
2) $\mathbf{S}$ is superfluous in $\mathbf{R}$ if and only if $f(\mathbf{S})$ is superfluous in $f(\mathbf{R})$.
3) The image of the core of $\mathbf{R}$, is the core of the image of $\mathbf{R}$, and the inverse image of the core of $f(\mathbf{R})$ is the core of the original image. That is, $Core(\mathbf{R})$ is the core of $\mathbf{R}$ if and only if $Core(f(\mathbf{R}))$ is the core of $f(\mathbf{R})$.

Remark 8: If f is just a general homomorphism from $(U, \mathbf{R})$ to $(V, f(\mathbf{R}))$, it can only keep the statement that S is a reduct

TABLE VII
RELATION $f(R_2)$

T-7	y_1	y_2
y_1	0.8	0.7
y_2	0.9	0.7

TABLE VIII
RELATION $f(R_1) \cap (R_2)$

T-8	y_1	y_2
y_1	0.6	0.7
y_2	0.6	0.7

TABLE IX
RELATION R_1

T-1	x1	x2	x3	x4	x5	x6	x7
x1	0.7	0.4	0.7	0.5	0.7	0.4	0.7
x2	0.7	0.3	0.5	0.8	0.5	0.3	0.7
x3	0.6	0.4	0.2	0.5	0.2	0.4	0.6
x4	0.6	0.3	0.5	0.8	0.5	0.3	0.6
x5	0.6	0.4	0.2	0.5	0.2	0.4	0.6
x6	0.7	0.3	0.5	0.8	0.5	0.3	0.7
x7	0.7	0.4	0.7	0.5	0.7	0.4	0.7

of $\mathbf{R} \Rightarrow f(\mathbf{S})$ is a reduct of $f(\mathbf{R})$ true. It cannot guarantee that the converse statement is true.

Parallel to Theorem 4.10, there is a similar theorem for a homomorphism from $(V, \mathbf{P})$ to $\left(U, f^{-1}(\mathbf{P})\right)$.

Theorem 4.12: Let $(V, \mathbf{P})$ be a relation information system, $\left(U, f^{-1}(\mathbf{P})\right)$ a f-induced relation information system of $(V, \mathbf{P})$, f^{-1} a homomorphism from $(V, \mathbf{P})$ to $\left(U, f^{-1}(\mathbf{P})\right)$ and $\mathbf{Q} \subseteq \mathbf{P}$. Then, $\mathbf{Q}$ is a reduct of $\mathbf{P}$ if and only if $f^{-1}(\mathbf{Q})$ is a reduct of $f^{-1}(\mathbf{P})$.

Proof: It is similar to the proof of Theorem 4.10. □

The following example is employed to illustrate our idea in Theorem 4.9 and 4.10.

Example 4.1: Let $(U, \mathbf{R})$ be a fuzzy relation information system, where $U = \{x_1, x_2, \ldots, x_7\}$, and $\mathbf{R} = \{R_1, R_2, R_3\}$. Let R_1, R_2, R_3 be as described in Tables IX–XI, respectively. Denote $R_1 \cap R_2 \cap R_3$ as in Table XII.

Let $V = \{y_1, y_2, y_3, y_4\}$. Define a mapping $f : U \rightarrow V$ as follows:

$$\frac{x_1, x_7 \qquad x_2, x_6 \qquad x_3, x_5 \qquad x_4}{y_1 \qquad\quad y_2 \qquad\quad y_3 \qquad\quad y_4}.$$

Then, $f(\mathbf{R}) = \{f(R_1), f(R_2), f(R_3)\}$, and $f(R_1)$, $f(R_2)$, $f(R_3)$ are expressed as Tables XIII–XV.

In addition, $(V, f(\mathbf{R}))$ is the f-induced fuzzy relation information system of $(U, \mathbf{R})$. It is very easy to verify that f is a compatible homomorphism from $(U, \mathbf{R})$ to $(V, f(\mathbf{R}))$.

We can see that $f(R_1)$ is superfluous if $f(\mathbf{R}) \Leftrightarrow R_1$ is superfluous in R and that $\{f(R_2), f(R_3)\}$, and is a reduct of $f(\mathbf{R}) \Leftrightarrow \{R_2, R_3\}$ is a reduct of $\mathbf{R}$. Therefore, we can reduce the original system by reducing the image system and reduce the image system by reducing the original system.

Definition 4.13: Let U be a finite universe of discourse, $\mathbf{R}$ a finite set of fuzzy T-similarity relations called conditional

TABLE X
RELATION R_2

T-2	x1	x2	x3	x4	x5	x6	x7
x1	0.4	0.5	0.7	0.5	0.7	0.5	0.4
x2	0.6	0.8	0.5	0.8	0.5	0.8	0.6
x3	0.7	0.9	0.2	0.9	0.2	0.9	0.7
x4	0.6	0.8	0.5	0.8	0.5	0.8	0.6
x5	0.7	0.9	0.2	0.9	0.2	0.9	0.7
x6	0.6	0.8	0.5	0.8	0.5	0.8	0.6
x7	0.4	0.5	0.7	0.5	0.7	0.5	0.4

TABLE XI
RELATION R_3

T-3	x1	x2	x3	x4	x5	x6	x7
x1	0.8	0.3	0.7	0.8	0.7	0.3	0.8
x2	0.7	0.2	0.6	0.7	0.6	0.2	0.7
x3	0.4	0.4	0.9	0.4	0.9	0.4	0.4
x4	0.8	0.3	0.7	0.8	0.7	0.3	0.8
x5	0.4	0.4	0.9	0.4	0.9	0.4	0.4
x6	0.7	0.2	0.6	0.7	0.6	0.2	0.7
x7	0.8	0.3	0.7	0.8	0.7	0.3	0.8

TABLE XII
RELATION $R_1 \cap R_2 \cap R_3$

T-4	x1	x2	x3	x4	x5	x6	x7
x1	0.4	0.3	0.7	0.5	0.7	0.3	0.4
x2	0.6	0.2	0.5	0.7	0.5	0.2	0.6
x3	0.4	0.4	0.2	0.4	0.2	0.4	0.4
x4	0.6	0.3	0.5	0.8	0.5	0.3	0.6
x5	0.4	0.4	0.2	0.4	0.2	0.4	0.4
x6	0.6	0.2	0.5	0.7	0.5	0.2	0.6
x7	0.4	0.3	0.7	0.5	0.7	0.3	0.4

TABLE XIII
RELATION $f(R_1)$

T-5	y1	y2	y3	Y4
y1	0.7	0.4	0.7	0.5
y2	0.7	0.3	0.5	0.8
y3	0.6	0.4	0.2	0.5
y4	0.6	0.3	0.5	0.8

attribute set, and D an equivalence relation called decision attribute with symbolic values. Then, $(U, \mathbf{R} \cup D)$ is called a T-fuzzy decision system. Denote $\text{Int}(\mathbf{R}) = \cap\{R : R \in \mathbf{R}\}$; then, $\text{Int}(\mathbf{R})$ is also a fuzzy T-similarity relation.

Definition 4.14: Let $U$and V be finite universes, $f : U \rightarrow V$ a mapping from U to V, and $(U, \mathbf{R} \cup D)$ a T-fuzzy decision system. Then, the pair $(V, f(\mathbf{R}) \cup f(D))$ is referred to as a f-induced T-fuzzy decision system of $(U, \mathbf{R} \cup D)$.

Definition 4.15: Let $(U, \mathbf{R} \cup D)$ be a T-fuzzy decision system, $x \in U, R \subseteq \mathbf{R}$, and $\mathbf{S} \subseteq \mathbf{R}$. $[x]_D$ is the equivalence class with respect to D and the positive region of D relative to $\text{Int}(\mathbf{R})$is defined as $\text{Pos}_{\text{Int}(\mathbf{R})} D = \cup_{x \in U} \underline{\text{Int}(\mathbf{R})_\vartheta}([x]_D)$. R is said to be superfluous with respect to D in $\mathbf{R}$ if $\text{Pos}_{\text{Int}(\mathbf{R})}$

TABLE XIV
RELATION $f(R_2)$

T-6	y1	y2	y3	Y4
y1	0.4	0.5	0.7	0.5
y2	0.6	0.8	0.5	0.8
y3	0.7	0.9	0.2	0.9
y4	0.6	0.8	0.5	0.8

TABLE XV
RELATION $f(R_3)$

T-7	y1	y2	y3	y4
y1	0.8	0.3	0.7	0.8
y2	0.7	0.2	0.6	0.7
y3	0.4	0.4	0.9	0.4
y4	0.8	0.3	0.7	0.8

$D = \mathrm{Pos}_{\mathrm{Int}(\mathbf{R}-R)}D$; otherwise, it is indispensable with respect to D. $\mathbf{S}$ is referred to as a reduct of $\mathbf{R}$ if $\mathbf{S}$ satisfies the following conditions.

1) $\mathrm{Pos}_{\mathrm{Int}(\mathbf{R})}D = \mathrm{Pos}_{\mathrm{Int}(\mathbf{S})}D$.
2) $\forall R_i \in \mathbf{S}, \mathrm{Pos}_{\mathrm{Int}(\mathbf{R})}D \supset \mathrm{Pos}_{\mathrm{Int}(\mathbf{S}\text{-}\{R_i\})}D$.

The collection of all indispensable elements with respect to D in $\mathbf{R}$ is called the core of $\mathbf{R}$ with respect to D, which is denoted as $Core_D(\mathbf{R})$.

Theorem 4.16: Let $(U, \mathbf{R} \cup D)$ be a T-fuzzy decision system, $(V, f(\mathbf{R}) \cup f(D))$ a f-induced T-fuzzy decision system, and fa compatible homomorphism from $(U, \mathbf{R} \cup D)$ to $(V, f(\mathbf{R}) \cup f(D))$, and $\mathbf{S} \subseteq \mathbf{R}$. Then, $\mathbf{S}$ is a reduct of $\mathbf{R}$ if and only if $f(\mathbf{S})$ is a reduct of $f(\mathbf{R})$.

Proof: $\Rightarrow$ Since

$$\underline{\mathrm{Int}(\mathbf{R})_\vartheta}[x]_D(y) = \underline{\underline{\mathrm{Int}(\mathbf{R})_\vartheta}}[x]_D(y) = \cup\left\{\overline{\mathrm{Int}(\mathbf{R})_T z_\lambda}(y) : \overline{\mathrm{Int}(\mathbf{R})_T z_\lambda} \subseteq [x]_D\right\}$$

we have

$$f\left(\underline{\mathrm{Int}(\mathbf{R})_\vartheta}[x]_D\right) = \cup\left\{f\left(\overline{\mathrm{Int}(\mathbf{R})_T z_\lambda}\right) : \overline{\mathrm{Int}(\mathbf{R})_T z_\lambda} \subseteq [x]_D\right\} = \cup\left\{\overline{f(\mathrm{Int}(\mathbf{R}))_T f(z)_\lambda} : \overline{\mathrm{Int}(\mathbf{R})_T z_\lambda} \subseteq [x]_D\right\}$$

by (1) of Theorem 3.12. Since f is a compatible homomorphism from $(U, \mathbf{R} \cup D)$ to $(V, f(\mathbf{R}) \cup \underline{f(D)})$, by Definition 4.3 and Theorem 3.12, it follows that $\overline{\mathrm{Int}(\mathbf{R})_T z_\lambda} \subseteq [x]_D \Leftrightarrow f\left(\overline{\mathrm{Int}(\mathbf{R})_T z_\lambda}\right) \subseteq f([x]_D)$. This implies

$$\begin{aligned} f\left(\underline{\mathrm{Int}(\mathbf{R})_\vartheta}[x]_D\right) &= \cup\left\{\overline{f(\mathrm{Int}(\mathbf{R}))f(z)_\lambda} : f\left(\overline{\mathrm{Int}(\mathbf{R})_T z_\lambda}\right) \subseteq f([x]_D)\right\} \\ &= \cup\left\{\overline{\mathrm{Int}(f(\mathbf{R}))_T f(z)_\lambda} : \overline{\mathrm{Int}(f(\mathbf{R}))_T f(z)_\lambda} \subseteq f([x]_D)\right\} \\ &= \underline{\underline{\mathrm{Int}(f(\mathbf{R}))}}_\vartheta f([x]_D) \\ &= \underline{\mathrm{Int}(f(\mathbf{R}))}_\vartheta f([x]_D) \\ &= \underline{\mathrm{Int}(f(\mathbf{R}))}_\vartheta [f(x)]_{f(D)} \end{aligned}$$

by (1) of Theorem 3.12. Since $\mathrm{Pos}_{\mathrm{Int}(\mathbf{R})}D = \cup_{x\in U}\underline{\mathrm{Int}(\mathbf{R})_\vartheta}([x]_D)$, we have

$$\begin{aligned} f\left(\mathrm{Pos}_{\mathrm{Int}(\mathbf{R})}D\right) &= f\left(\cup_{x\in U}\underline{\mathrm{Int}(\mathbf{R})_\vartheta}([x]_D)\right) \\ &= \cup_{x\in U} f\left(\underline{\mathrm{Int}(\mathbf{R})_\vartheta}([x]_D)\right) \\ &= \cup_{x\in U}\underline{\mathrm{Int}(f(\mathbf{R}))}_\vartheta [f(x)]_{f(D)}. \end{aligned}$$

Similar to the above reasoning, we have

$$\begin{aligned} f\left(\mathrm{Pos}_{\mathrm{Int}(\mathbf{S})}D\right) &= f\left(\cup_{x\in U}\underline{\mathrm{Int}(\mathbf{S})_\vartheta}([x]_D)\right) \\ &= \cup_{x\in U} f\left(\underline{\mathrm{Int}(\mathbf{S})_\vartheta}([x]_D)\right) \\ &= \cup_{x\in U}\underline{\mathrm{Int}(f(\mathbf{S}))}_\vartheta [f(x)]_{f(D)} \end{aligned}$$

for $\mathbf{S} \subseteq \mathbf{R}$. Since $\mathbf{S}$ is a reduct of $\mathbf{R}$, it follows that $\mathrm{Pos}_{\mathrm{Int}(\mathbf{R})}D = \mathrm{Pos}_{\mathrm{Int}(\mathbf{S})}D$, which implies $f\left(\mathrm{Pos}_{\mathrm{Int}(\mathbf{R})}D\right) = f\left(\mathrm{Pos}_{\mathrm{Int}(\mathbf{S})}D\right)$. Hence, $\mathrm{Pos}_{\mathrm{Int}(f(\mathbf{R})))}f(D) = \mathrm{Pos}_{\mathrm{Int}(f(\mathbf{S}))} f(D)$.

Assume that there is a fuzzy T-similarity relation R_i such that $\mathrm{Pos}_{\mathrm{Int}(f(\mathbf{S}))}f(D) = \mathrm{Pos}_{\mathrm{Int}(f(\mathbf{S}\text{-}\{R_i\}))}f(D)$; then, we have $\mathrm{Pos}_{\mathrm{Int}(f(\mathbf{R}))}f(D) = \mathrm{Pos}_{\mathrm{Int}(f(\mathbf{S}-\{R_i\}))}f(D)$, i.e.,

$$\cup_{x\in U}\underline{\mathrm{Int}(f(\mathbf{R}))}_\vartheta [f(x)]_{f(D)} = \cup_{x\in U}\underline{\mathrm{Int}(f(\mathbf{S}-R_i))}_\vartheta [f(x)]_{f(D)}.$$

Since $[f(x)]_{f(D)} \cap [f(y)]_{f(D)} = \oslash$ and $[f(x)]_{f(D)} \neq [f(y)]_{f(D)}$, we have $\underline{\mathrm{Int}(f(\mathbf{R}))}_\vartheta [f(x)]_{f(D)} = \underline{\mathrm{Int}(f(\mathbf{S}-R_i))}_\vartheta [f(x)]_{f(D)}$ for every $x \in U$. Since f is a compatible homomorphism from $(U, \mathbf{R} \cup D)$ to $(V, f(\mathbf{R}) \cup f(D))$, similar to the above proof, we have $f\left(\underline{\mathrm{Int}(\mathbf{R})_\vartheta}[x]_D\right) = f\left(\underline{\mathrm{Int}(\mathbf{S}-R_i)_\vartheta}[x]_D\right)$, which implies that $\underline{\mathrm{Int}(\mathbf{R})_\vartheta}[x]_D = \underline{\mathrm{Int}(\mathbf{S}-R_i)_\vartheta}[x]_D$. Hence, $\mathrm{Pos}_{\mathrm{Int}(\mathbf{R})}D = \mathrm{Pos}_{\mathrm{Int}(\mathbf{S}-\{R_i\})}D$, which is a contradiction to the fact that $\mathbf{S}$ is a reduct of $\mathbf{R}$.

$\Leftarrow$ Similar to the above proof, and we can get the conclusion.

Corollary 4.17: Let $(V, f(\mathbf{R}) \cup f(D))$ be a f-induced fuzzy relation information system of $(U, \mathbf{R} \cup D)$, $R \in \mathbf{R}$, and $\mathbf{S} \subseteq \mathbf{R}$. If f is a compatible homomorphism from $(U, \mathbf{R})$ to $(V, f(\mathbf{R}))$. Then we have the following.

1) R is indispensable with respect to D in $\mathbf{R}$ if and only if $f(R)$ is indispensable in$f(\mathbf{R})$ with respect to $f(D)$.
2) $\mathbf{S}$ is superfluous with respect to D in $\mathbf{R}$ if and only if $f(\mathbf{S})$ is superfluous in $f(\mathbf{R})$ with respect to $f(D)$.
3) The image of the relative core of $\mathbf{R}$ is the relative core of the image of $\mathbf{R}$ and the inverse image of the relative core of $f(\mathbf{R})$ is the relative core of the original image. That is, $Core_D(\mathbf{R})$ is the core of $\mathbf{R}$ if and only if $Core_{f(D)}(f(\mathbf{R}))$ is the core of $f(\mathbf{R})$.

Remark 9: If f is just a general homomorphism from $(U, \mathbf{R})$ to $(V, f(\mathbf{R}))$, it can only keep the statement that $\mathbf{S}$ is a relative reduct of $\mathbf{R} \Rightarrow f(\mathbf{S})$ if a relative reduct of $f(\mathbf{R})$ is true. It cannot guarantee that the converse statement is true.

V. Conclusion

In this paper, we have study the communication between information systems with fuzzy relations. We point out that a mapping between two universes can induce a fuzzy binary relation of one universe according to the given fuzzy relation on the other universe. For an information system with fuzzy relations, we can consider it as a combination of some fuzzy approximation spaces on the same universe. The mapping between fuzzy approximation spaces can be explained as a mapping between the given fuzzy relation information systems. Based on these observations, we have explored properties of fuzzy relation mappings and discussed the characteristics of fuzzy relation information systems and found that attribute reductions of the original system and its image system are equivalent to each other under homomorphism.

These results illustrate that some characteristics of a fuzzy relation information system are guaranteed in its image system, which may have potential applications in knowledge reduction, decision making, and reasoning about data, especially for the case of two information systems. These results will also help in forming a systematic and theoretic framework for information-granule communication analysis. However, some results in this paper are right only under some sufficient conditions. How to explore the sufficient and necessary conditions of these results is our future work.

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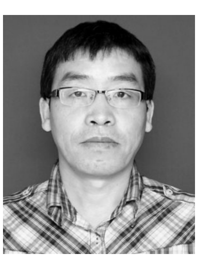

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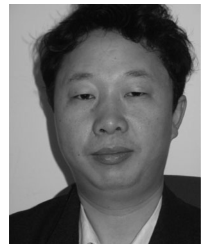

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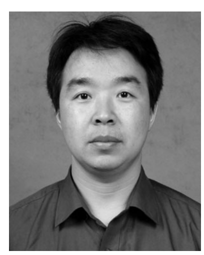

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