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# Communication between information systems with covering based rough sets

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ABSTRACT

Communication between information systems is considered as an important issue in granular computing. The concept of homomorphism is an effective mathematical tool to study information exchange between information systems. This paper provides a study on some basic properties of covering information systems and decision systems under homomorphisms. First, we define consistent functions related to coverings and covering mappings between two universes, and study their properties. Then, we introduce the notions of homomorphisms of covering information systems and point out that a homomorphism is a special covering mapping between information systems. Furthermore, we investigate some important properties of homomorphisms in covering information systems and decision systems. It is proved that some basic properties of original systems, such as set approximations, attribute reductions, can be reserved under the condition of homomorphisms in both covering information systems and covering decision systems.


## 1. Introduction

The theory of rough sets [28–31], proposed by Pawlak, is a mathematical tool to deal with inexact or uncertain knowledge in information systems. This theory can approximate subsets of universes by two definable subsets called lower and upper approximations and unravel knowledge hidden in information systems [8–10,14,21–32,35,36,45,55–61]. Another application of rough set theory is to reduce the number of attributes in databases. Given a dataset with discrete attribute values, we can find a subset of attributes that are the most informative and has the same discernible capability as the original attributes [15,16,18,19,37–43,50–54].

Information system, as a mathematical model in artificial intelligence, is an important application area of rough sets. Over the last decades, there has been much work on information systems with rough sets, including some successful applications in machine learning, pattern recognition, decision analysis, and knowledge discovery in databases. These topics may be divided broadly into two classes: One is to investigate internally the information mining and processing in information systems. The other is to study information transmission or mapping between information systems, which is so called the communication between information systems [1,11–13,17,20,33,34,47–49,62].

Communication is directly related to the issue of mappings of information systems while preserving their basic functions. The original motivation to study communication between information systems is to find a relatively small database which

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has the same results on reduction as the original database [1,12,13,20]. This topic can be extended to many important application fields such as data fusion, data compression [49], and information tranmision [11,34]. According to ideas in [11,33,34], communication can be explained as translating the information contained in one granular world into the granularity of another granular world and thus providing a mechanism for exchanging information with other granular worlds. From mathematical viewpoints, these communications can be considered as to compare some structures and properties of different information systems via mappings, which are useful tools to study the relationship between information systems.

The theory of rough sets combined with homomorphic mapping concepts in algebras is a new strategy to study communication between information systems. According to the viewpoint in rough sets, a rough approximation space is actually a granular information world and an information system can be seen as a combination of some approximation spaces on the same universe. Thus a mapping between approximation spaces can induce a mapping between information systems and communication between information systems can be explained as a mapping between two information systems [12,13,20,33,34]. The notion of homomorphism based on rough sets, as a special mapping between information systems, was firstly introduced by Grzymala-Busse in [12,13]. Later, Li and Ma discussed some features of redundancy and reductions of complete information systems under some homomorphisms [20]. As showed in these works, the notion of homomorphism on information systems is useful in aggregating sets of objects, attributes, and descriptors of original systems. However, these two studies are mainly concentrated on the problem about attribute reductions under homomorphisms. They did not discuss the issue related to set approximations. Furthermore, their researches are both limited in the framework of traditional rough sets.

As we know, traditional rough set model works in case that the values of attributes are only symbolic. In reality, the values of attributes could be both symbolic and real-valued in information systems. Thus the notions of homomorphisms based on traditional rough sets cannot be applied into studying commication between information systems containing real-valued data. To deal with this problem, several consistent functions based on binary relations were introduced and investigated in [47,49]. In [48], Wang further defined the notions of consistent functions based on fuzzy relations for constructing attribute reducts and examining some invariant properties of homomorphisms in fuzzy environments.

The theory of covering based rough sets [60] is another important generalization of classical rough sets to deal with information systems based on coverings which is so called covering information system. This theory has been amply demonstrated to be useful by successful applications in a variety of problems [2–7,9,26,35,36,44,46,63–66]. Recently, Wang et al. introduced the concepts of hommorphisms with covering based rough sets in order to construct attribute reducts of covering information systems in communications[51]. A problem with Wang's study is that many important issues were not discussed. For example, sets approximations and attribute reductions in covering decision systems have not been considered and the necessary conditions of some hommorphic properties of covering information systems were also not presented.

The article makes some new contributions to the development of the theory of communications between information systems and between decision systems. We firstly introduce the concepts of consistent functions in covering information systems. Later, we respectively define the concepts of homomorphisms between covering information systems and between covering decision systems. Under the condition of homomorphisms, we discuss some properties of covering information systems and decision systems, and present some relationships of structural features of original systems and their image systems. We find that some basic properties of original systems, such as set approximations, attribute reductions, can be reserved under the condition of homomorphisms in both covering information systems and covering decision systems.

The remainder of this paper is organized as follows. In Section 2, we review the relevant concepts in rough set theory. In Section 3, we present the definitions of consistent functions related to coverings and investigate their properties. In Section 4, we define the concepts of covering mappings between two universes and investigate their properties. In Section 5, we introduce the concepts of homomorphisms between covering information systems and study their properties. In Section 6, we present the concepts of homomorphisms between covering decision systems and study their properties. Section 7 presents conclusions.

## 2. Preliminaries

This section mainly reviews some basic notions related to this paper.

### 2.1. Pawlak's rough sets

An information system is a pair $\mathbf{A} = (U,A)$, where $U = \{x_1,\ldots,x_n\}$ is a nonempty and finite set of objects and $A = \{a_1,a_2,\ldots,a_m\}$ is a nonempty and finite set of attributes. For any $B \subseteq A$ we associate a binary relation $Ind(B)$, called $B$-indiscernibility relation, and defined as

$$Ind(B) = \{(x,y) \in U \times U : a(x) = a(y), \forall a \in B\}.$$

Obviously, $Ind(B)$ is an equivalence relation and $Ind(B) = \cap_{a\in B} Ind(\{a\})$. By $[x]_B$ we denote the equivalence class including $x$ with respect to $Ind(B)$. For any subset $X \subseteq U$, let

$$\underline{B}X = \{x \in U : [x]_B \subseteq X\}, \quad \overline{B}X = \{x \in U : [x]_B \cap X \neq \emptyset\}.$$

Then $\underline{B}(X)$ and $\underline{B}(X)$ are called lower and upper approximations of $X$ with respect to $Ind(B)$, respectively.

An attribute $a \in B$ is superfluous in $B$ if $Ind(B) = Ind(B - \{a\})$, otherwise $a$ is indispensable in $B$. The set of all indispensable attributes in $A$ is called the core of $(U,A)$, denoted as $Core(B)$. Let $B \subseteq A$. $B$ is called independent in $A$ if each attribute in $B$ is indispensable in $B$. $B$ is called a reduct of $(U,A)$ if $B$ is independent and $Ind(B) = Ind(A)$.

A decision system is a pair $\mathbf{A}^* = (U, A \cup \{d\})$, where $d$ is a decision attribute, $A$ is the condition attribute set. Let $U/\{d\}$ denote the set of all equivalence classes induced by $d$. Let $Pos_B(d) = \cup_{X \in U/d} \underline{B}X$. Then $Pos_B(d)$ is called the positive region of $d$. We say $a \in A$ is relatively dispensable in $B$ if $Pos_B(d) = Pos_{B-\{a\}}(d)$, otherwise $a$ is said to be relatively indispensable in $B$. Let $B \subseteq A$. If each attribute in $B$ is relatively indispensable in $B$, we say that $B$ is relatively independent in $A$. $B$ is called a relative reduct of $\mathbf{A}^*$ if $B$ is relatively independent in $A$ and $Pos_B(d) = Pos_A(d)$.

### 2.2. Basic notions related to covering based rough sets

In this subsection, we briefly review some basic concepts related to covering information systems.

**Definition 2.1.** Let $U$ be a universe of discourse, $\mathbf{C}$ a family of subsets of $U$. $\mathbf{C}$ is called a covering of $U$ if no subset in $\mathbf{C}$ is empty and $\cup\mathbf{C} = U$.

It is clear that a partition of $U$ is certainly a covering of $U$ and the concept of a covering is an extension of a partition. In [35,36] the notion of coverings was used to construct lower and upper approximation operators and to study properties of these operators. In the following, we review the concepts of neighborhoods induced by coverings and the corresponding approximation operators to be used in following sections [7,56,63–65].

**Definition 2.2.** Let $\mathbf{C} = \{K_1, K_2, \ldots, K_n\}$ be a covering of $U$. For every $x \in U$, let $C_x = \cap\{K_i : x \in K_i \wedge K_i \in \mathbf{C}\}$ and $Cov(\mathbf{C}) = \{C_x : x \in U\}$, then $C_x$ is a neighborhood of $x$ induced by $\mathbf{C}$ and $Cov(\mathbf{C})$ is a covering on $U$. We call $Cov(\mathbf{C})$ the induced covering of $\mathbf{C}$.

For any $x \in U$, $C_x$ is the minimal subset including $x$ in $Cov(\mathbf{C})$, every element in $Cov(\mathbf{C})$ cannot be written as the union of other elements in $Cov(\mathbf{C})$. $Cov(\mathbf{C}) = \mathbf{C}$ if and only if $\mathbf{C}$ is a partition. For any $x, y \in U$, if $y \in C_x$ then $C_y \subseteq C_x$. So if $y \in C_x$ and $x \in C_y$, then $C_x = C_y$. The relationships between elements in $Cov(\mathbf{C})$ have the following properties.

(1) Reflexivity: $\forall x \in U$, $x \in C_x$.
(2) Anti-symmetry: if $y \in C_x$ and $x \in C_y$, then $C_x = C_y$.
(3) Transitivity: $\forall x, y, z \in U$, if $x \in C_y$ and $y \in C_z$, then $x \in C_z$.

**Definition 2.3.** Let $\mathbf{\Delta} = \{\mathbf{C}_i : i = 1, \ldots, m\}$ be a family of coverings of $U$. For every $x \in U$, let $\Delta_x = \cap\{C_{ix} : C_{ix} \in Cov(\mathbf{C}_i), i = 1, 2, \ldots, m\}$ and $Cov(\Delta) = \{\Delta_x : x \in U\}$. Then $\Delta_x$ is a neighborhood of $x$ with respect to $\Delta$ and $Cov(\Delta)$ is a covering on $U$. We call $Cov(\Delta)$ the induced covering of $\Delta$.

Clearly $\Delta_x$ is the intersection of all covering elements including $x$ in all coverings in $\mathbf{\Delta}$. So for every $x \in U$, $\Delta_x$ is the minimal subset including $x$ in $Cov(\mathbf{\Delta})$ and $Cov(\mathbf{\Delta})$ can be viewed as the intersection of coverings in $\mathbf{\Delta}$. Every element in $Cov(\mathbf{\Delta})$ cannot be written as the union of other elements in $Cov(\mathbf{\Delta})$. If every covering in $\mathbf{\Delta}$ is a partition, then $Cov(\mathbf{\Delta})$ is also a partition and $\Delta_x$ is the equivalence class including $x$. For any $x, y \in U$, if $y \in \Delta_x$, then $\Delta_y \subseteq \Delta_x$. So if $y \in \Delta_x$ and $x \in \Delta_y$, then $\Delta_x = \Delta_y$. Therefore, the relationships between elements in $Cov(\mathbf{\Delta})$ also have such properties as reflexivity, anti-symmetry and transitivity.

In [56] Yao introduced a pair of approximation operators based on the concept of neighborhoods. Let $n$: $U \to 2^U$ be an arbitrary neighborhood operator and $n(x)$ the corresponding neighborhood of $x$. The approximation operators are defined as follows:

$$\underline{apr}_n(X) = \{x \in U | n(x) \subseteq X\}, \overline{apr}_n(X) = \{x \in U | n(x) \cap X \neq \emptyset\}.$$

If $n(x)$ is replaced by the neighborhood of $x$ induced by coverings, we can define a particular approximation operators [7,63–65].

**Definition 2.4.** Let $\Delta = \{\mathbf{C}_i : i = 1, 2, \ldots, m\}$ a family of $U$ and $X \subseteq U$. The upper and lower approximation of $X$ relative to $Cov(\mathbf{\Delta})$ are respectively defined as follows: $\underline{apr}_\Delta X = \{x \in U : \Delta_x \subseteq X\}$, $\overline{apr}_\Delta X = \{x \in U : \Delta_x \cap X \neq \emptyset\}$.

The positive domain, negative domain and boundary domain of $X$ relative to $\mathbf{\Delta}$ are respectively computed by the following formulas: $Pos_\Delta(X) = \underline{apr}_\Delta X$, $Neg_\Delta(X) = U - \overline{apr}_{\mathbf{\Delta}} X$, $Bn_\Delta(X) = \overline{apr}_{\mathbf{\Delta}} X - \underline{apr}_{\mathbf{\Delta}} X$.

## 3. Consistent functions and their properties

Mapping is a basic mathematical tool to study the relationship between two sets. Similarly, we study communication between covering information systems by mapping. In order to study invariant properties of covering information systems in communications, we firstly introduce the concepts of consistent functions with respect to coverings and investigate their basic properties in this section. Let $U$ be a nonempty set called a universe. We denote by $C(U)$ the set of all coverings on $U$.

**Definition 3.1.** Let $U$ and $V$ be universes, $f$: $U \to V$ a mapping, $\mathbf{C} = \{K_1, K_2, \ldots, K_n\}$ a covering on $U$, and $Cov(\mathbf{C}) = \{C_x : x \in U\}$. The mapping $f$ is called a consistent function with respect to $\mathbf{C}$ if for any $x, y \in U$, $C_x = C_y$ whenever $f(x) = f(y)$. Let $X \subseteq U$, then $f$ is called a consistent function with respect to $X$ if $x \in X$ implies that $y \in X$ for any $x, y \in U$ satisfying $f(x) = f(y)$.

From Definition 3.1, an injection is trivially a consistent function with respect to a covering and an arbitrary subset of $U$.

To illustrate the above definition, let us see a simple example.

**Example 3.1.** Let $U = \{x_1, x_2, \ldots, x_9\}$, $V = \{y_1, y_2, \ldots, y_5\}$. $\mathbf{C} = \{K_1, K_2, K_3\}$ is a covering on $U$, where $K_1 = \{x_1, x_2, x_4, x_8\}$, $K_2 = \{x_3, x_4, x_6, x_7, x_8, x_9\}$ and $K_3 = \{x_3, x_5, x_7\}$. Then $C_{x_1} = C_{x_2} = K_1$, $C_{x_3} = C_{x_7} = \{x_3, x_7\}$, $C_{x_4} = C_{x_8} = \{x_4, x_8\}$, $C_{x_5} = K_3$, $C_{x_6} = C_{x_9} = K_2$. $Cov(\mathbf{C}) = \{C_{x_i} : i = 1, 2, \ldots, 9\}$. Define a mapping $f$: $U \to V$ as follows:

$$\frac{x_1, x_2 \quad x_3, x_7 \quad x_4, x_8 \quad x_6, x_9 \quad x_5}{y_1 \qquad\quad y_2 \qquad\quad y_3 \qquad y_4 \qquad y_5}.$$

It is verified that $f$ is a consistent function with respect to $\mathbf{C}$.

It is clear that a consistent function with respect to a covering is a special mapping between two universes. This class of functions has many good properties which can be used to construct the notions of homomorphisms in subsequent sections. In the following we examine some basic properties of a consistent function.

**Proposition 3.1.** *Let $U$ and $V$ be universal sets, $f$: $U \to V$ a mapping, $\mathbf{C} = \{K_1, K_2, \ldots, K_n\}$ a covering on $U$, $Cov(\mathbf{C}) = \{C_x : x \in U\}$. Then $f$ is consistent with respect to $\mathbf{C}$ if and only if for any $K_i \in \mathbf{C}$, $f$ is consistent with respect to $K_i$.*

**Proof.** It follows immediately from Definition 3.1.

The next theorem extends the assertion of Theorem 3.2 in [51], where only the sufficiency has been provided. □

**Theorem 3.1.** *Let $U$ and $V$ be nonempty universal sets, $\mathbf{C} = \{K_1, K_2, \ldots, K_n\}$ a covering on $U$, and $f$: $U \to V$ a mapping. Then for any $K_i \in \mathbf{C}$, $f^{-1}(f(K_i)) = K_i$ if and only if $f$ is a consistent function with respect to $\mathbf{C}$.*

**Proof.** $\Rightarrow$Suppose that there exist $x_0, x \in U$ with $f(x_0) = f(x)$ such that $C_{x_0} \neq C_x$. By the reflexivity of $C_x$ we have $x \in C_x$, which implies $f(x) \in f(C_x)$. It follows from $f(x_0) = f(x)$ that $f(x_0) \in f(C_x)$, and thus $x_0 \in f^{-1}(f(C_x))$. Without loss of generality, suppose that $x \in K_{i_k}, k \leqslant m < n$. Then $C_x \subseteq K_{i_k}$. Let $K_{i_k} = C_x \cup \{K_{i_k} - C_x\}$. Then $f^{-1}(f(K_{i_k})) = f^{-1}(f(C_x \cup \{K_{i_k} - C_x\})) = f^{-1}(f(C_x)) \cup f^{-1}(f(K_{i_k} - C_x))$. Since $f^{-1}(f(K_{i_k})) = K_{i_k}$, we have that $f^{-1}(f(C_x)) \subseteq K_{i_k}$ for any $k \leqslant m < n$. This, together with $x_0 \in f^{-1}(f(C_x))$, implies that $x_0 \in \bigcap_{k=1}^{m} K_{i_k} = C_x$, namely, $x_0 \in C_x$, which means $C_{x_0} \subseteq C_x$ by the transitivity of $C_x$. Similarly, we can get that $C_x \subseteq C_{x_0}$. This forces that $C_{x_0} = C_x$, a contradiction. Whence, $f$ is a consistent function with respect to $\mathbf{C}$.

$\Leftarrow$ We only need to show that $f^{-1}(f(K_i)) \subseteq K_i$ since the inverse inclusion is always true. For any $x \in f^{-1}(f(K_i))$, then $f(x) \in f(K_i)$. Thus there exists $y \in K_i$ such that $f(x) = f(y)$. This, together with the definition of $C_y$, have that $C_y \subseteq K_i$. As $f$ is a consistent function with respect to $\mathbf{C}$, we have that $C_x = C_y$ by the fact $f(x) = f(y)$. It follows from the reflexivity of $C_x$ that $x \in C_x = C_y$. This implies by $C_y \subseteq K_i$ that $x \in K_i$,as desired. □

**Remark.** According to the result of this theorem, we also can say that a function $f$ defined on $U$ is to be called a consistent function with respect to a covering $\mathbf{C}$ if and only if for any $K_i \in \mathbf{C}$, $f^{-1}(f(K_i)) = K_i$.

From Proposition 3.1 and Theorem 3.1, we immediately get the following corollary.

**Corollary 3.1.** *Let $U$ and $V$ be nonempty universal sets, $\mathbf{C} = \{K_1, K_2, \ldots, K_n\}$ a covering on $U$, and $f$: $U \to V$ a mapping. Then $f$ is a consistent function with respect to $\mathbf{C}$ if and only if for any $x \in U$, $f^{-1}(f(C_x)) = C_x$.*

*Suppose that $\mathbf{C} = \{K_1, K_2, \ldots, K_n\}$ a covering on $U$, and $f$: $U \to V$ a mapping. Theorem 3.1 in [51] shows us that if $f$ is consistent with respect to $\mathbf{C}$, then $f(K_i \cap K_j) = f(K_i) \cap f(K_j)$. In fact, the next theorem shows us that this equatity holds just when $f$ is consistent with respect to either $K_i$ or $K_j$.*

**Theorem 3.2.** *Let $U$ and $V$ be nonempty universal sets, $\mathbf{C} = \{K_1, K_2, \ldots, K_n\}$ a covering on $U$, and $f$: $U \to V$ a mapping. For any $K_i$, $K_j \in \mathbf{C}$ $(i, j \leqslant n)$, if $f$ is consistent with respect to either $K_i$ or $K_j$, then $f(K_i \cap K_j) = f(K_i) \cap f(K_j)$.*

**Proof.** Without loss of generality, we may assume that $f$ is consistent with respect to $K_i$. At first**,** we are to prove that $f(K_i) \cap f(K_j) = \emptyset$ if $K_i \cap K_j = \emptyset$. By contradiction, assume that $f(K_i) \cap f(K_j) \neq \emptyset$. Suppose $u \in f(K_i) \cap f(K_j)$, then $u \in f(K_i)$ and $u \in f(K_j)$. Hence, there exist $x \in K_i$ and $y \in K_j$ such that $f(x) = f(y) = u$, which implies that $y \in K_i$ since $f$ is consistent with respect to $K_i$. We thus have that $y \in K_i \cap K_j$, a contradiction. Hence $f(K_i) \cap f(K_j) = \emptyset$. Next, we are to prove that if $K_i \cap K_j \neq \emptyset$, then $f(K_i \cap K_j) = f(K_i) \cap f(K_j)$. We only need to prove $f(K_i \cap K_j) \supseteq f(K_i) \cap f(K_j)$ since the inverse inclusion is always true. For any $u \in f(K_i) \cap f(K_j)$, there exist $x \in K_i$ and $y \in K_j$ such that $f(x) = f(y) = u$, which implies that $y \in K_i$ since $f$ is consistent with respect to $K_i$. We thus have that $y \in K_i \cap K_j$. Hence, $f(y) \in f(K_i \cap K_j)$. This, together with $u = f(y)$, gives rise to $u \in f(K_i \cap K_j)$, as desired. □

**Corollary 3.2.** *Let U and V be nonempty universal sets,* $\mathbf{C} = \{K_1, K_2, \ldots, K_n\}$ *a covering on U, and* $f: U \to V$ *a mapping. If f is consistent with respect to* $\mathbf{C}$*, then* $f(C_x) = \cap\{f(K_i):K_i \in \mathbf{C} \wedge x \in K_i\}$ *for any* $x \in U$*.*

**Proof.** It follows immediately from Proposition 3.1 and Theorem 3.2. □

In order to better understand invariant properties of intersection and union operations of some coverings in communication between different universes, we introduce some basic concepts related to covering operations in the following. Let us start with introducing the notion of order relation between two coverings.

**Definition 3.2.** Let $U$ be a nonempty universe and $\mathbf{C}_1, \mathbf{C}_2 \in C(U)$, where $\mathbf{C}_1 = \{K_{11}, K_{12}, \ldots, K_{1m}\}$, $\mathbf{C}_2 = \{K_{21}, K_{22}, \ldots, K_{2n}\}$. For any $K_{1k} \in \mathbf{C}_1 (k \leqslant m)$, if there always exists $K_{2l} \in \mathbf{C}_2 (l \leqslant n)$ such that $K_{1k} \subseteq K_{2l}$, then $\mathbf{C}_1$ is called contained in $\mathbf{C}_2$, denoted as $\mathbf{C}_1 \leqslant \mathbf{C}_2$.

**Remark.** If $\mathbf{C}_1 \leqslant \mathbf{C}_2$ and $\mathbf{C}_1 \geqslant \mathbf{C}_2$, we cannot conclude that $\mathbf{C}_1 = \mathbf{C}_2$ is necessarily true.

The following definition characterizes the intersection and union operations of coverings on a universe.

**Definition 3.3.** Let $U$ be a nonempty universe and $\mathbf{C}_1, \mathbf{C}_2 \in C(U)$, where $\mathbf{C}_1 = \{K_{11}, K_{12}, \ldots, K_{1m}\}$, $\mathbf{C}_2 = \{K_{21}, K_{22}, \ldots, K_{2n}\}$. Let

$$\mathbf{C}_1 \cap \mathbf{C}_2 = \{C_{1x} \cap C_{2x} : C_{ix} \in Cov(\mathbf{C}_i), i = 1, 2, x \in U\}, \quad \mathbf{C}_1 \cup \mathbf{C}_2 = \{K_{1i} : i \leqslant m\} \cup \{K_{2j} : j \leqslant n\}.$$

Then $\mathbf{C}_1 \cap \mathbf{C}_2$ is called the intersection of $\mathbf{C}_1$ and $\mathbf{C}_2$ and $\mathbf{C}_1 \cup \mathbf{C}_2$ is called the union of $\mathbf{C}_1$ and $\mathbf{C}_2$.

From Definition 3.3, for every $x \in U$, $C_{1x} \cap C_{2x}$ is the minimal subset including $x$ in $\mathbf{C}_1 \cap \mathbf{C}_2$. Clearly $\mathbf{C}_1 \cap \mathbf{C}_2$ and $\mathbf{C}_1 \cup \mathbf{C}_2$ are both coverings of $U$.

The following theorem shows some intuitive properties of operations of coverings.

**Theorem 3.3.** *Let U be a nonempty universal set and* $\mathbf{C}_1, \mathbf{C}_2 \in C(U)$*. Then*

(1) $\mathbf{C}_1 \cap \mathbf{C}_2 = Cov(\mathbf{C}_1) \cap Cov(\mathbf{C}_2)$;
(2) $\mathbf{C}_1 \cap \mathbf{C}_2 = Cov(\mathbf{C}_1 \cup \mathbf{C}_2)$;
(3) $\mathbf{C}_1 \cap \mathbf{C}_2 \leqslant \mathbf{C}_i \leqslant \mathbf{C}_1 \cup \mathbf{C}_2$.

**Proof.** Straightforward. □

The following theorem presents the equivalence statement of consistent functions related to intersection and union operations.

**Theorem 3.4.** *Let* $f: U \to V$ *and* $\mathbf{C}_1, \mathbf{C}_2 \in C(U)$*. Then the following statements are equivalent:*

(1) *f is consistent with respect to* $\mathbf{C}_1$ *and* $\mathbf{C}_2$ *respectively;*
(2) *f is consistent with respect to* $\mathbf{C}_1 \cup \mathbf{C}_2$*;*
(3) *f is consistent with respect to* $\mathbf{C}_1 \cap \mathbf{C}_2$*.*

**Proof.** For $x \in U$, let $C_x$ be the minimal subset including $x$ in $\mathbf{C}_1 \cap \mathbf{C}_2$, $C'_x$ the minimal subset including $x$ in $Cov(\mathbf{C}_1 \cup \mathbf{C}_2)$, $C_{1x}$ the minimal subset including $x$ in $Cov(\mathbf{C}_1)$ and $C_{2x}$ the minimal subset including $x$ in $Cov(\mathbf{C}_2)$. It follows from Definitions 2.2 and 3.3 that $C'_x = \cap\{K : x \in K, K \in \mathbf{C}_1 \cup \mathbf{C}_2\}$ and so $C'_x = C_{1x} \cap C_{2x}$. From Definition 3.3, we have that $C_{1x} \cap C_{2x} = C_x$, and so $C'_x = C_x$. Similarly, we have that $C_{1y} \cap C_{2y} = C_y$ and $C'_y = C_y$ for any $y \in U$.

(1) ⇒(2) Since $f$ is consistent with respect to $\mathbf{C}_1$ and $\mathbf{C}_2$ respectively, it follows that $C_{1x} = C_{1y}$ and $C_{2x} = C_{2y}$ for any $x, y \in U$ satisfying $f(x) = f(y)$. Thus $C_{1x} \cap C_{2x} = C_{1y} \cap C_{2y}$, which implies by $C_{1x} \cap C_{2x} = C_x$ and $C_{1y} \cap C_{2y} = C_y$ that $C_x = C_y$. This, together with the fact $C'_x = C_x$, and $C'_y = C_y$, means that $C'_x = C'_y$. Hence by the fact $f(x) = f(y)$ we know that $f$ is consistent with respect to $\mathbf{C}_1 \cup \mathbf{C}_2$.
(2) ⇒(3) Since $f$ is consistent with respect to $\mathbf{C}_1 \cup \mathbf{C}_2$, it follows from the Definitions 3.1 and 3.3 that $C'_x = C'_y$ for any $x, y \in U$ satisfying $f(x) = f(y)$. By the fact that $C'_x = C_x$ and $C'_y = C_y$ we get that $C_x = C_y$. Then we know from Definition 3.1 that $f$ is consistent with respect to $\mathbf{C}_1 \cap \mathbf{C}_2$.
(3) ⇒(1) Since $f$ is consistent with respect to $\mathbf{C}_1 \cap \mathbf{C}_2$, it follows that $C_x = C_y$ for any $x, y \in U$ satisfying $f(x) = f(y)$. This implies by the fact that $C_{1x} \cap C_{2x} = C_x$ and $C_{1y} \cap C_{2y} = C_y$ that $C_{1x} \cap C_{2x} = C_{1y} \cap C_{2y}$. It follows from the reflexivity and transitivity of $C_{1x}$, $C_{2x}$, $C_{1y}$, $C_{2y}$ that $C_{1x} = C_{1y}$ and $C_{2x} = C_{2y}$. Thus by the fact $f(x) = f(y)$ we know that $f$ is consistent with respect to $\mathbf{C}_1$ and $\mathbf{C}_2$, respectively. □

**Corollary 3.3.** *Let U and V be nonempty universal sets, $\mathbf{\Delta} = \{\mathbf{C}_i: i = 1,2,\ldots,m\}$ a family of coverings on U, and $f: U \to V$ a mapping. Then $f^{-1}(f(\Delta_x)) = \Delta_x$ for any $x \in U$ if and only if f is consistent with respect to each $\mathbf{C}_i \in \Delta$.*

**Proof.** It follows immediately from Definitions 2.3 and 3.3, and Theorems 3.1 and 3.4. □

## 4. Covering mappings and their properties

In order to develop tools for studying communication between information systems with covering based rough sets, this section is devoted to introduce covering mappings and explore their properties. Let us first review the extension principle of classical sets.

Let $f$ be a mapping from $U$ to $V$, $f: U \to V$, $u| \to f(u) = v \in V$, $\forall u \in U$. The extension principle of classical sets shows that $f$ can induce a mapping from the power set $P(U)$ to the power set $P(V)$ and an inverse mapping from $P(V)$ to $P(U)$, that is,

$$f: P(U) \to P(V), \quad A| \to f(A) = \{v \in V | v = f(u), u \in A\}, \quad \forall A \in P(U);$$
$$f^{-1}: P(V) \to P(U), \quad B| \to f^{-1}(B) = \{u \in U | f(u) \in B\}, \quad \forall B \in P(V).$$

where $f(A)$ is referred to as the image of $A$ and $f^{-1}(B)$ is referred to as the inverse image of $B$.

The extension principle of classical sets extends the concepts of mappings from classical sets to power sets. Based on the idea of the extension principle, we introduce the concepts of covering mappings as follows.

**Definition 4.1.** Let $f: U \to V$, $x| \to f(x)$. $f$ can induce a mapping from $C(U)$ to $C(V)$ and a mapping from $C(V)$ to $C(U)$, that is,

$$\hat{f}: C(U) \to C(V), \quad \mathbf{C}| \to f(\mathbf{C}) \in C(V), \quad \forall \mathbf{C} \in C(U); \quad \hat{f}(\mathbf{C}) = \{f(K_i): K_i \in \mathbf{C}\};$$
$$\hat{f}^{-1}: C(V) \to C(U), \quad \mathbf{E}| \to f^{-1}(\mathbf{E}) \in C(U), \quad \forall \mathbf{E} \in C(V); \quad \hat{f}^{-1}(\mathbf{E}) = \{f^{-1}(L_i): L_i \in \mathbf{E}\}.$$

Then $\hat{f}$ and $\hat{f}^{-1}$ are called covering mapping and inverse covering mapping induced by $f$ respectively; $\hat{f}(\mathbf{C})$ and $\hat{f}^{-1}(\mathbf{E})$ are called the image of $\mathbf{C}$ and the inverse image of $\mathbf{E}$, respectively. Under no confusion, we simply denote $\hat{f}$ and $\hat{f}^{-1}$ by $f$ and $f^{-1}$, respectively.

To illustrate the above definition, let us see the following example.

**Example 4.1.** Let $U = \{x_1,x_2,\ldots,x_9\}$ and $V = \{y_1,y_2,\ldots,y_5\}$. $\mathbf{C} = \{K_1,K_2,K_3\}$ is a covering on $U$, where $K_1 = \{x_1,x_7,x_8\}$, $K_2 = \{x_2,x_3,x_6,x_7,x_9\}$, $K_3 = \{x_3,x_4,x_5,x_6\}$. Define a mapping $f: U \to V$ as follows:

$$\frac{x_1,x_8 \quad x_2,x_9 \quad x_3,x_6 \quad x_4,x_5 \quad x_7}{y_1 \qquad y_2 \qquad y_3 \qquad y_4 \qquad y_5}.$$

Then $f(K_1) = \{y_1,y_5\}$, $f(K_2) = \{y_2,y_3,y_5\}$, $f(K_3) = \{y_3,y_4\}$. Thus $f(\mathbf{C}) = \{f(K_1),f(K_2),f(K_3)\}$. Let $\mathbf{C}' = \{K'_1,K'_2,K'_3\}$ be another covering on $V$, where $K'_1 = \{y_1,y_2\}$, $K'_2 = \{y_2,y_3,y_4\}$, $K'_3 = \{y_4,y_5\}$. Then $f^{-1}(K'_1) = \{x_1,x_2,x_8,x_9\}$, $f^{-1}(K'_2) = \{x_2,x_3,x_4,x_5,x_6,x_9\}$, $f^{-1}(K'_3) = \{x_4,x_5,x_7\}$. Thus $f^{-1}(\mathbf{C}') = \{f^{-1}(K'_1),f^{-1}(K'_2),f^{-1}(K'_3)\}$.□

In the following, we investigate basic properties of covering mappings.

The following theorem shows us that if $f$ is a consistent function with respect to one of two coverings, then $f$ can preserve the intersection of covering elements.

**Theorem 4.1.** *Let $f: U \to V$ and $\mathbf{C}_1, \mathbf{C}_2 \in C(U)$. If f is a consistent function with respect to either $\mathbf{C}_1$ or $\mathbf{C}_2$, then $f(C_{1x} \cap C_{2x}) = f(C_{1x}) \cap f(C_{2x})$ for any $x \in U$.*

**Proof.** We are only to prove that $f(C_{1x} \cap C_{2x}) \supseteq f(C_{1x}) \cap f(C_{2x})$ since the inverse inclusion is always true. Without loss of generality, we may assume that $f$ is consistent with respect to $\mathbf{C}_1$. Let $z \in f(C_{1x}) \cap f(C_{2x})$, then there exist $y_1 \in C_{1x}$ and $y_2 \in C_{2x}$ such that $f(y_1) = f(y_2) = z$. This implies that $C_{1y} \subseteq C_{1x}$ by the transitivity of $C_{1x}$ and $C_{1_{y_1}} = C_{1_{y_2}}$ by the fact that $f$ is consistent with respect to $\mathbf{C}_1$. Thus $C_{1_{y_2}} \subseteq C_{1x}$. This, together with the reflexivity of $C_{1_{y_2}}$, gives rise to that $y_2 \in C_{1x}$. Hence we have that $y_2 \in C_{1x} \cap C_{2x}$, which implies that $z = f(y_2) \in f(C_{1x} \cap C_{2x})$. Consequently, $f(C_{1x}) \cap f(C_{2x}) \subseteq f(C_{1x} \cap C_{2x})$. This, together with the fact that $C_x = C_{1x} \cap C_{2x}$, means that $f(C_{1x}) \cap f(C_{2x}) \subseteq f(C_x)$, as desired.

The following theorem clarifies the relationship between a covering and its image or inverse image. It shows that a covering could make itself change in communication between universes if a mapping is not restrained. □

**Theorem 4.2.** *Let $\mathbf{C} \in C(U)$ and $\mathbf{E} \in C(V)$. Then*

(1) *$f(f^{-1}(\mathbf{E})) \leqslant \mathbf{E}$; then the equality holds if and only if f is surjective.*
(2) *$f^{-1}(f(\mathbf{C})) \geqslant \mathbf{C}$; the equality holds if and only if f is a consistent function with respect to $\mathbf{C}$ on U.*

**Proof.**

(1) Straightforward.
(2) Let $K_i$ be arbitrary subset in $\mathbf{C}$, i.e., $K_i \in \mathbf{C}$, then $f^{-1}(f(K_i)) \supseteq K_i$. Thus $f^{-1}(f(\mathbf{C})) \geqslant \mathbf{C}$. It follows from Theorem 3.1 that $f^{-1}(f(K_i)) = K_i$ for any $K_i \in \mathbf{C}$ if and only if $f$ is a consistent function with respect to $\mathbf{C}$. Hence $f^{-1}(f(\mathbf{C})) = \mathbf{C}$ if and only if $f$ is a consistent function with respect to $\mathbf{C}$. □

**Corollary 4.1.** *Let $f$: $U \to V$, $\mathbf{\Delta} = \{\mathbf{C}_i : i = 1,\ldots,m\}$ and $\Gamma = \{\mathbf{E}_1, \mathbf{E}_2,\ldots,\mathbf{E}_n\}$. Then*

(1) *$f(f^{-1}(\cap\Gamma)) \leqslant \cap\Gamma$; then the equality holds if and only if $f$ is surjective.*
(2) *$f^{-1}(f(\cap\Delta)) \geqslant \cap\Delta$; the equality holds if and only if for any $\mathbf{C}_i \in \Delta$, $f$ is a consistent function with respect to $\mathbf{C}_i$.*

**Proof.** Follows immediately from Theorems 3.4 and 4.2(2). □

Let $f$: $U \to V$, $\mathbf{C} = \{K_1, K_2,\ldots,K_n\}$ a covering of $U$ and $Cov(\mathbf{C}) = \{C_x : x \in U\}$, where $C_x$ can be seen as the information granule of $x$ with respect to $\mathbf{C}$ and $Cov(\mathbf{C})$ can be viewed as a set of information granules with respect to $\mathbf{C}$ on $U$. For any $x \in U$, let

$$f(C)_{f(x)} = \cap\{f(K_i) : f(x) \in f(K_i) \wedge f(K_i) \in f(\mathbf{C})\}, \quad Cov(f(\mathbf{C})) = \{f(C)_{f(x)} : x \in U\}.$$

Similarly, $f(C)_{f(x)}$ can be seen as the information granule of $f(x)$ with respect to $f(\mathbf{C})$ and $Cov(f(\mathbf{C}))$ can be viewed as a set of information granules with respect to $f(\mathbf{C})$ on $V$.

The following theorem says that the form of an information granule with respect to a covering is kept unchanged in communication between two universes, when a mapping is consistent with respect to the covering. That is, the image of the information granule related to an object in an original system is the information granule related to the image of the object in its image system.

**Theorem 4.3.** *Let $f$: $U \to V$, $\mathbf{C} = \{K_1, K_2,\ldots,K_n\}$ a covering of $U$. If $f$ is a consistent function with respect to $\mathbf{C}$, then*
*(1) $f(C_x) = f(C)_{f(x)}$; (2) $f(Cov(\mathbf{C})) = Cov(f(\mathbf{C}))$.*

**Proof.**

(1) We are to complete the proof by the following three steps.
  (i) If $f$ is consistent with respect to $\mathbf{C}$, then $x \in K_i \Leftrightarrow f(x) \in f(K_i)$.
  Let $x \in K_i$, then $f(x) \in f(K_i)$. Conversely, if $f(x) \in f(K_i)$, then $x \in f^{-1}(f(x)) \subseteq f^{-1}(f(K_i))$. Since $f$ is consistent with respect to $\mathbf{C}$, it follows from Theorem 3.1, that $f^{-1}(f(K_i)) = K_i$. Hence $x \in K_i$.
  (ii) If $f$ is consistent with respect to $\mathbf{C}$, then $K_i \in \mathbf{C} \Leftrightarrow f(K_i) \in f(\mathbf{C})$.
  It obviously true that $K_i \in \mathbf{C} \Rightarrow f(K_i) \in f(\mathbf{C})$. Conversely, if $f(K_i) \in f(\mathbf{C})$, then $f^{-1}(f(K_i)) \subseteq f^{-1}(f(\mathbf{C}))$. Since $f$ is consistent with respect to $\mathbf{C}$, it follows from Theorems 3.1 and 4.2(2) that $f^{-1}(f(K_i)) = K_i$ and $f^{-1}(f(\mathbf{C})) = \mathbf{C}$. Hence $K_i \in \mathbf{C}$, i.e., $f(K_i) \in f(\mathbf{C}) \Rightarrow K_i \in \mathbf{C}$.
  (iii) Next, we are to prove $f(C_x) = f(C)_{f(x)}$. Since $f$ is consistent with respect to $\mathbf{C}$ and $C_x = \cap\{K_i : x \in K_i \wedge K_i \in \mathbf{C}\}$, it follows from Corollary 3.2 that $f(C_x) = \cap\{f(K_i) : x \in K_i \wedge K_i \in \mathbf{C}\}$. From (i) and (ii), we have that $f(C_x) = \cap\{f(K_i) : f(x) \in f(K_i) \wedge f(K_i) \in f(\mathbf{C})\}$. Hence $f(C_x) = f(C)_{f(x)}$.
(2) Follows immediately from (1). □

Let $f$: $U \to V$, $\mathbf{\Delta} = \{\mathbf{C}_i : i = 1,\ldots,m\}$ a family of coverings of $U$ and $Cov(\mathbf{\Delta}) = \{\Delta_x : x \in U\}$. Similarly, $\Delta_x$ can be viewed as the information granule of $x$ with respect to $\mathbf{\Delta}$ and $f(\Delta)_{f(x)}$ can be viewed as the information granule of $f(x)$ with respect to $f(\Delta)$. Similar to Theorem 4.3, we have the same results for an information granule related to a family of coverings in communication between different universes.

**Theorem 4.4.** *Let $f$: $U \to V$, $\mathbf{\Delta} = \{\mathbf{C}_i : i = 1,\ldots,m\}$ a family of coverings of $U$ and $Cov(\mathbf{\Delta}) = \{\Delta_x : x \in U\}$. For any $x \in U$, let $f(\Delta)_{f(x)} = \cap\{f(C_i)_{f(x)} : f(\mathbf{C}_i) \in f(\Delta)\}$ and $Cov(f(\Delta)) = \{f(\Delta)_{f(x)} : x \in U\}$. If $f$ is a consistent function with respect to each covering $\mathbf{C}_i \in \Delta$, then (1) $f(\Delta_x) = f(\Delta)_{f(x)}$; (2) $f(Cov(\Delta)) = Cov(f(\Delta))$.*

**Proof.**

(1) Since $\Delta_x = C_{1x} \cap C_{2x} \cap \cdots \cap C_{mx}$, it follows that $f(\Delta_x) \subseteq \bigcap_{i=1}^m f(C_{ix})$. For any $y \in \bigcap_{i=1}^m f(C_{ix})$, we have $y \in f(C_{ix})$, $i = 1, 2,\ldots,m$. Thus there exist $u_1 \in C_{1x}$, $u_2 \in C_{2x}$, ..., $u_m \in C_{mx}$ such that $f(u_1) = f(u_2) = \cdots = f(u_m) = y$. Since $f$ is consistent respect to each covering $\mathbf{C}_i \in \Delta$, it follows from Definition 3.1 and Theorem 3.4 that $u_m \in \bigcap_{i=1}^m C_{ix} = \Delta_x$. Thus $y = f(u_m) \in f(\bigcap_{i=1}^m C_{ix}) = f(\Delta_x)$. Hence $\bigcap_{i=1}^m f(C_{ix}) \subseteq f(\Delta_x)$. So $f(\Delta_x) = \bigcap_{i=1}^m f(C_{ix})$. Since $f$ is consistent with respect to $\mathbf{C}_i$ ($i \leqslant m$), by Theorem 4.3(1), we have that $f(\Delta_x) = \bigcap_{i=1}^m f(C_{ix}) = \bigcap_{i=1}^m f(C_i)_{f(x)}$. Therefore $f(\Delta_x) = f(\Delta)_{f(x)}$.
(2) Follows immediately from (1). □

A mapping can keep the order relation of coverings. For the union operation, any mapping preserves coverings while coverings can be kept unchanged for intersection operation only if a mapping is consistent with these coverings.

**Theorem 4.5.** *Let $f$: $U \to V$ and $\mathbf{C}_1$, $\mathbf{C}_2 \in C(U)$. Then*

(1) $\mathbf{C}_1 \leqslant \mathbf{C}_2 \Rightarrow f(\mathbf{C}_1) \leqslant f(\mathbf{C}_2)$;
(2) $f(\mathbf{C}_1 \cup \mathbf{C}_2) = f(\mathbf{C}_1) \cup f(\mathbf{C}_2)$;
(3) $f(\mathbf{C}_1 \cap \mathbf{C}_2) \leqslant f(\mathbf{C}_1) \cap f(\mathbf{C}_2)$; *If f is a consistent function with respect to either* $\mathbf{C}_1$ *or* $\mathbf{C}_2$, *then the equality holds.*

**Proof.** (1) and (2) are obviously true.

(3) The first part of this assertion is evidently true. We only prove the second part. For any $x \in U$, let $C_x$ be the minimal element containing $x$ in $\mathbf{C}_1 \cap \mathbf{C}_2$, $C_{1x}$ the minimal element containing $x$ in $Cov(\mathbf{C}_1)$, $C_{2x}$ the minimal element containing $x$ in $Cov(\mathbf{C}_2)$. From Definition 3.3, we get $C_x = C_{1x} \cap C_{2x}$. It follows from Theorem 4.1 that $f(C_x) = f(C_{1x}) \cap f(C_{2x})$. Since $f$ is consistent with respect to $\mathbf{C}_1$ or $\mathbf{C}_2$, it follows from Theorem 4.3(1) that $f(C_{1x}) = f(C_1)_{f(x)}$ and $f(C_{2x}) = f(C_2)_{f(x)}$. Thus $f(C_x) = f(C_1)_{f(x)} \cap f(C_2)_{f(x)}$. By Definition 3.3, we know that $f(C_1)_{f(x)} \cap f(C_2)_{f(x)}$ is an element including $f(x)$ in $f(\mathbf{C}_1) \cap f(\mathbf{C}_2)$. This implies that for any $x \in U$, $f(C_1)_{f(x)} \cap f(C_2)_{f(x)} = f(C_x) \in f(\mathbf{C}_1 \cap \mathbf{C}_2)$. Hence $f(\mathbf{C}_1 \cap \mathbf{C}_2) \geqslant f(\mathbf{C}_1) \cap f(\mathbf{C}_2)$, as desired. □

**Remark.** In general, a covering mapping $f$ does not keep invariant the intersection of coverings.

**Corollary 4.2.** *Let $f$: $U \to V$ and $\mathbf{\Delta} = \{\mathbf{C}_i : i = 1,\ldots,m\}$ a family of coverings of U. Then*

(1) $f\left(\bigcup_{i=1}^{n} \mathbf{C}_i\right) = \bigcup_{i=1}^{n} f(\mathbf{C}_i)$;
(2) If $f$ is consistent respect to any $\mathbf{C}_i \in \Delta$, then $f\left(\bigcap_{i=1}^{n} \mathbf{C}_i\right) = \bigcap_{i=1}^{n} f(\mathbf{C}_i)$.

**Proof.** It follows immediately from Theorems 3.4 and 4.5(2). □

Suppose that $f$: $U \to V$ is mapping, $\mathbf{E} = \{L_1, L_2, \ldots, L_n\}$ is a covering of $V$ and $Cov(\mathbf{E}) = \{E_u : u \in V\}$. For any $u \in U$, $E_u$ can be viewed as the information granule of $u$ with respect to $\mathbf{E}$ and $f^{-1}(E)_x$ can be viewed as the information granule of $x$ with respect to $f^{-1}(\mathbf{E})$, where $f(x) = u$. The following theorem shows that the inverse image of the information granule of an object in an image system is the information granule of an inverse image of the object in its original system.

**Theorem 4.6.** *Let $f$: $U \to V$, $\mathbf{E} = \{L_1, L_2, \ldots, L_n\}$ a covering of V and $Cov(\mathbf{E}) = \{E_u : u \in V\}$. For any $x \in U$, let*

$$f^{-1}(E)_x = \cap\{f^{-1}(L_i) : x \in f^{-1}(L_i) \wedge f^{-1}(L_i) \in f^{-1}(\mathbf{E})\}, \quad Cov(f^{-1}(\mathbf{E})) = \{f^{-1}(E)_x : x \in U\}.$$

*If f is surjective, then (1) $f^{-1}(E_{f(x)}) = f^{-1}(E)_x$; (2) $f^{-1}(Cov(\mathbf{E})) = Cov(f^{-1}(\mathbf{E}))$.*

**Proof.**

(1) Since $f$ is surjective, we have by Theorem 4.2 that $x \in f^{-1}(L_i) \Leftrightarrow f(x) \in L_i$ and $L_i \in \mathbf{E} \Leftrightarrow f^{-1}(L_i) \in f^{-1}(\mathbf{E})$. Thus $f^{-1}(E)_x = \cap\{f^{-1}(L_i) : x \in f^{-1}(L_i) \wedge f^{-1}(L_i) \in f^{-1}(\mathbf{E})\} = f^{-1}(\cap\{L_i : f(x) \in L_i \wedge L_i \in \mathbf{E}\}) = f^{-1}(E_{f(x)})$.
(2) Follows immediately from (1). □

For communication between different universes, an inverse mapping can preserve intersection and union operations of coverings, and also can keep the order relation between coverings.

**Theorem 4.7.** *Let $f$: $U \to V$ and $\mathbf{E}_1$, $\mathbf{E}_2 \in C(V)$. Then*

(1) $\mathbf{E}_1 \leqslant \mathbf{E}_2 \Rightarrow f^{-1}(\mathbf{E}_1) \subseteq f^{-1}(\mathbf{E}_2)$;
(2) $f^{-1}(\mathbf{E}_1 \cup \mathbf{E}_2) = f^{-1}(\mathbf{E}_1) \cup f^{-1}(\mathbf{E}_2)$;
(3) *If f is surjective, then* $f^{-1}(\mathbf{E}_1 \cap \mathbf{E}_2) = f^{-1}(\mathbf{E}_1) \cap f^{-1}(\mathbf{E}_2)$.

**Proof.** (1) and (2) are obviously true.

(3) For any $u \in V$, let $E_u$ be the minimal subset containing $u$ in $\mathbf{E}_1 \cap \mathbf{E}_2$, $E_{1u}$ the minimal subset containing $u$ in $Cov(\mathbf{E}_1)$ and $E_{2u}$ the minimal subset containing $u$ in $Cov(\mathbf{E}_2)$. Let $x \in U$ such that $f(x) = u$. Let $C_x$ be the minimal subset containing $x$ in $f^{-1}(\mathbf{E}_1) \cap f^{-1}(\mathbf{E}_2)$. Since $f$ is surjective, it follows from Theorem 4.6(2) that $f^{-1}(Cov(\mathbf{E}_1)) = Cov(f^{-1}(\mathbf{E}_1))$. So $E_{1u} \in Cov(\mathbf{E}_1) \Longleftrightarrow f^{-1}(E_{1u}) \in Cov(f^{-1}(\mathbf{E}_1))$. Similarly, $E_{2u} \in Cov(\mathbf{E}_2) \Longleftrightarrow f^{-1}(E_{2u}) \in Cov(f^{-1}(\mathbf{E}_2))$. By Definition 2.3, $E_u = \cap\{E_{iu} : E_{iu} \in Cov(\mathbf{E}_i), i = 1, 2\}$. Thus $E_u = (\cap\{E_{1u} : E_{1u} \in Cov(\mathbf{E}_1)\}) \cap (\cap\{E_{2u} : E_{2u} \in Cov(\mathbf{E}_2)\})$. Hence $f^{-1}(E_u) = f^{-1}(\cap\{E_{1u} : E_{1u} \in Cov(\mathbf{E}_1)\}) \cap f^{-1}(\cap\{E_{2u} : E_{2u} \in Cov(\mathbf{E}_2)\})$. It follows that $f^{-1}(E_u) = (\cap\{f^{-1}(E_{1u}) : f^{-1}(E_{1u}) \in Cov(f^{-1}(\mathbf{E}_1))\}) \cap (\cap\{f^{-1}(E_{2u}) : f^{-1}(E_{2u}) \in Cov(f^{-1}(\mathbf{E}_2))\}) = C_x$. This means that $f^{-1}(E_u) \in f^{-1}(\mathbf{E}_1) \cap f^{-1}(\mathbf{E}_2)$. So $f^{-1}(\mathbf{E}_1 \cap \mathbf{E}_2) \subseteq f^{-1}(\mathbf{E}_1) \cap f^{-1}(\mathbf{E}_2)$.

Conversely, for any $x \in U$, let $C_x$ be the minimal subset containing $x$ in $f^{-1}(\mathbf{E}_1) \cap f^{-1}(\mathbf{E}_2)$. Let $f(x) = u \in V$, then $E_u = \cap\{E_{iu} : E_{iu} \in Cov(\mathbf{E}_i), i = 1, 2\} \in \mathbf{E}_1 \cap \mathbf{E}_2$. This implies that $f^{-1}(E_u) \in f^{-1}(\mathbf{E}_1 \cap \mathbf{E}_2)$. It follows from the fact $C_x = f^{-1}(E_u)$ that $C_x \in f^{-1}(\mathbf{E}_1 \cap \mathbf{E}_2)$. Hence $f^{-1}(\mathbf{E}_1) \cap f^{-1}(\mathbf{E}_2) \subseteq f^{-1}(\mathbf{E}_1 \cap \mathbf{E}_2)$. This completes the proof. □

The following theorem discusses the problems of rough approximations of a subset in a universe under covering mapping and inverse covering mapping, respectively.

**Theorem 4.8.** *Let $f$: $U \to V$, $\mathbf{C} \in C(U)$, $\mathbf{E} \in C(V)$, $X \subseteq U$ and $Y \subseteq V$. Then*

(1) *If $f$ is a consistent function with respect to $\mathbf{C}$, then $f(\underline{apr}_{\mathbf{C}}\, X) \subseteq \underline{apr}_{f(\mathbf{C})} f(X)$.*
(2) *Let $X$ be a $\mathbf{C}$-definable subset. If $f$ is surjective and consistent with respect to $\mathbf{C}$, then $f(\underline{apr}_{\mathbf{C}}\, X) = \underline{apr}_{f(\mathbf{C})} f(X) = f(X)$, namely, $f(X)$ is a $f(\mathbf{C})$-definable subset.*
(3) *If $f$ is a consistent function with respect to $\mathbf{C}$, then $f(\overline{apr}_{\mathbf{C}} X) \subseteq \overline{apr}_{f(\mathbf{C})} f(X)$.*
(4) *Let $X$ be a $\mathbf{C}$-definable subset. If $f$ is surjective and consistent with respect to $\mathbf{C}$, then $f(\overline{apr}_{\mathbf{C}} X) = \overline{apr}_{f(\mathbf{C})} f(X) = f(X)$, namely, $f(X)$ is a $f(\mathbf{C})$-definable subset.*
(5) *If $f$ is surjective, then $f^{-1}(\underline{apr}_{\mathbf{E}} Y) = \underline{apr}_{f^{-1}(\mathbf{E})} f^{-1}(Y)$.*
(6) *If $f$ is surjective, then $f^{-1}(\overline{apr}_{\mathbf{E}} Y) = \overline{apr}_{f^{-1}(\mathbf{E})} f^{-1}(Y)$.*

**Proof.**

(1) For any $y \in f(\underline{apr}_{\mathbf{C}}\, X)$, there must exist $x \in \underline{apr}_{\mathbf{C}} X$ such that $f(x) = y$. By the definition of $\underline{apr}_{\mathbf{C}} X$, we have $C_x \subseteq X$. Thus $f(C_x) \subseteq f(X)$. Since $f$ is a consistent function with respect to $\mathbf{C}$, by Theorem 4.3, we have $f(C)_{f(x)} = f(C_x) \subseteq f(X)$. Thus $f(x) \in \underline{apr}_{f(\mathbf{C})} f(X)$. So $f(\underline{apr}_{\mathbf{C}}\, X) \subseteq \underline{apr}_{f(\mathbf{C})} f(X)$.
(2) From the result of (1), we only need to prove that $f(\underline{apr}_{\mathbf{C}}\, X) \supseteq \underline{apr}_{f(\mathbf{C})} f(X)$. For any $y \in \underline{apr}_{f(\mathbf{C})} f(X)$, we have $f(C)_y \subseteq f(X)$. Since $X$ is $\mathbf{C}$-definable subset, there is no loss of generality in assuming that $X = \cup C_u$. Since $f$ is consistent with respect to $\mathbf{C}$, it follows from Corollary 3.1 that $f^{-1}(f(C)_y) \subseteq f^{-1}(f(X)) = \cup C_u = X$. Let $x \in U$ such that $f(x) = y$, then $f^{-1}(f(C)_{f(x)}) \subseteq X$. Since $f$ is consistent with respect to $\mathbf{C}$, we have $f(C)_{f(x)} = f(C_x)$ by Theorem 4.3(1). This implies by Corollary 3.1 that $C_x = f^{-1}(f(C_x)) = f^{-1}(f(C)_{f(x)}) \subseteq X$. This means $x \in \underline{apr}_{\mathbf{C}} X$. Thus $y = f(x) \in f(\underline{apr}_{\mathbf{C}}\, X)$. Hence $f(\underline{apr}_{\mathbf{C}}\, X) \supseteq \underline{apr}_{f(\mathbf{C})} f(X)$.
(3) Let $y \in f(\overline{apr}_{\mathbf{C}} X)$, then there exists $x \in \overline{apr}_{\mathbf{C}} X$ such that $f(x) = y$. By the definition of $\overline{apr}_{\mathbf{C}} X$, we have $C_x \cap X \neq \emptyset$. Thus $f(C_x \cap X) \neq \emptyset$. Since $f(C_x) \cap f(X) \supseteq f(C_x \cap X)$, we know $f(C_x) \cap f(X) \neq \emptyset$. Since $f$ is consistent with respect to $\mathbf{C}$, by Theorem 4.3(1) we have $f(C_x) = f(C)_{f(x)}$. Thus $f(C)_{f(x)} \cap f(X) \neq \emptyset$, hence $y = f(x) \in \overline{apr}_{f(\mathbf{C})} f(X)$. So $f(\overline{apr}_{\mathbf{C}} X) \subseteq \overline{apr}_{f(\mathbf{C})} f(X)$.
(4) Since $X$ is $\mathbf{C}$-definable subset and $f$ is consistent with respect to $\mathbf{C}$, it follows that

$$f(\overline{apr}_{\mathbf{C}} X) = f(\sim \underline{apr}_{\mathbf{C}} \sim X) = \sim f(\underline{apr}_{\mathbf{C}} \sim X) \supseteq \sim \underline{apr}_{f(\mathbf{C})} f(\sim X) = \sim \underline{apr}_{f(\mathbf{C})} \sim f(X) = \overline{apr}_{f(\mathbf{C})} X.$$

By the result of (3), we know that it is true.

(5) Let $x \in f^{-1}(\underline{apr}_{\mathbf{E}}\, Y)$, then $f(x) \in \underline{apr}_{\mathbf{E}} Y$. Thus $E_{f(x)} \subseteq Y$, which implies $f^{-1}(E_{f(x)}) \subseteq f^{-1}(Y)$. It follows from Theorem 4.5(1) that $f^{-1}(E_{f(x)}) = f^{-1}(E)_x$. Thus $f^{-1}(E)_x \subseteq f^{-1}(Y)$. Hence $x \in \underline{apr}_{f^{-1}(\mathbf{E})} f^{-1}(Y)$. So $f^{-1}(\underline{apr}_{\mathbf{E}} Y) \subseteq \underline{apr}_{f^{-1}(\mathbf{E})} f^{-1}(Y)$. Conversely, let $x \in \underline{apr}_{f^{-1}(\mathbf{E})} f^{-1}(Y)$, then $f^{-1}(E)_x \subseteq f^{-1}(Y)$. Thus $f(f^{-1}(E)_x) \subseteq f(f^{-1}(Y)) \subseteq Y$. By Theorem 4.5(1), we have that $f^{-1}(E)_x = f^{-1}(E_{f(x)})$. Since $f$ is surjective, it follows that $f(f^{-1}(E)_x) = f(f^{-1}(E_{f(x)})) = E_{f(x)}$. This means that $E_{f(x)} \subseteq Y$. It follows that $f(x) \in \underline{apr}_{\mathbf{E}} Y$. Hence $x \in f^{-1}(x) \subseteq f^{-1}(\underline{apr}_{\mathbf{E}}\, Y)$. So $f^{-1}(\underline{apr}_{\mathbf{E}} Y) \supseteq \underline{apr}_{f^{-1}(\mathbf{E})} f^{-1}(Y)$, as desired.
(6) Follows from (5) and the duality of the approximation operators. □

## 5. Homomorphisms between covering information systems and their properties

As mentioned in Introduction, communication from one information system to another one can be regarded as homomorphism between information systems from mathematical viewpoints. By means of the results of the above sections, we introduce the notions of homomorphisms as tools to study communication between covering information systems, and examine its some invariant properties.

Let $f$: $U \to V$ be a surjective mapping from $U$ to $V$, $\Delta = \{\mathbf{C}_i : i = 1, 2, \ldots, m\}$ a family of coverings on $U$ and $\Gamma = \{\mathbf{E}_1, \mathbf{E}_2, \ldots, \mathbf{E}_n\}$ a family of coverings on $V$. Denote $f(\Delta) = \{f(\mathbf{C}_1), f(\mathbf{C}_2) \cdots f(\mathbf{C}_m)\}$ and $f^{-1}(\Gamma) = \{f^{-1}(\mathbf{E}_1), f^{-1}(\mathbf{E}_2) \cdots f^{-1}(\mathbf{E}_n)\}$. Let us start with introducing the notions of covering information systems.

**Definition 5.1.** Let $f$: $U \to V$ and $\Delta = \{\mathbf{C}_i : i = 1, 2, \ldots, m\}$ be a family of coverings on $U$. Then the pair $(U, \Delta)$ is referred to as a covering information system.

**Definition 5.2.** Let $f$: $U \to V$ be a surjective mapping from $U$ to $V$, and $\Gamma = \{\mathbf{E}_1, \mathbf{E}_2, \ldots, \mathbf{E}_n\}$ a family of coverings on $V$. Then the pair $(V, \Gamma)$ is referred to as a covering information system.

By Theorems 4.5, 4.7, we introduce the following concepts.

**Definition 5.3.** Let $(U,\Delta)$ be a covering information system. If $f$ is consistent with respect to each $\mathbf{C}_i \in \Delta$, then $f$ is referred to as a homomorphism on $(U,\Delta)$.

**Definition 5.4.** Let $(U,\Delta)$ be a covering information system. If $f$ is surjective, then $f^{-1}$ is referred to as a homomorphism from $(V,\Gamma)$ to $(U,f^{-1}(\Gamma))$.

**Remark.** After the notions of homomorphisms are introduced, all the theorems and corollaries in which the equality holds in the above sections may be viewed as the properties of homomorphisms.

**Definition 5.5.** Let $(U,\Delta)$ be a covering information system and $\mathbf{C}_i \in \mathbf{\Delta}$. Then $\mathbf{C}_i$ is called superfluous in $\mathbf{\Delta}$ if $\cap\{\Delta - \{\mathbf{C}_i\}\} = \cap\Delta$; otherwise, $\mathbf{C}_i$ is called indispensable in $\mathbf{\Delta}$. Let $\mathbf{P} \subseteq \Delta$, then $\mathbf{P}$ is referred to as a reduct of $\mathbf{\Delta}$ if $\mathbf{P}$ satisfies that $\cap\mathbf{P} = \cap\Delta$ and $\cap\mathbf{P} \neq \cap\{\mathbf{P} - \{\mathbf{C}_i\}\}$ for any $\mathbf{C}_i \in \mathbf{P}$. The collection of all indispensable elements in $\mathbf{\Delta}$ is called the core of $\mathbf{\Delta}$, denoted as $Core(\Delta)$.

**Theorem 5.1.** *Let $(U,\Delta)$ be a covering information system, $\mathbf{P} \subseteq \Delta$. If $f$ is a homomorphism on $(U,\Delta)$, Then $\mathbf{P}$ is a reduct of $\Delta$ if and only if $f(\mathbf{P})$ is a reduct of $f(\Delta)$.*

**Proof.** $\Rightarrow$Since $\mathbf{P}$ is a reduct of $\Delta$, we have $\cap\mathbf{P} = \cap\Delta$. Hence $f(\cap\mathbf{P}) = f(\cap\Delta)$. Since $f$ is a homomorphism on $(U,\Delta)$, by Definition 5.3 and Corollary 4.2(2), we have $\cap f(\mathbf{P}) = \cap f(\Delta)$. Assume that there exists $\mathbf{C}_i \in \Delta$ such that $\cap\{f(\mathbf{P}) - \{f(\mathbf{C}_i)\}\} = \cap f(\mathbf{P})$. Because $f(\mathbf{P}) - \{f(\mathbf{C}_i)\} = f(\mathbf{P} - \{\mathbf{C}_i\})$, we have that $\cap\{f(\mathbf{P}) - \{f(\mathbf{C}_i)\}\} = \cap f(\mathbf{P} - \{\mathbf{C}_i\}) = \cap f(\mathbf{P}) = \cap f(\Delta)$. Similarly, by Definition 5.3 and Corollary 4.2(2), it follows that $f(\cap\{\mathbf{P} - \{\mathbf{C}_i\}\}) = f(\cap\Delta)$. Thus $f^{-1}(f(\cap\{\mathbf{P} - \{\mathbf{C}_i\}\})) = f^{-1}(f(\cap\Delta))$. By Definition 5.3 and Corollary 4.1(2), $\cap\{\mathbf{P} - \{\mathbf{C}_i\}\} = \cap\Delta$. This is a contradiction to that $\mathbf{P}$ is a reduct of $\Delta$.

$\Leftarrow$ Let $f(\mathbf{P})$ be a reduct of $f(\Delta)$, then $\cap f(\mathbf{P}) = \cap f(\Delta)$. Since $f$ a homomorphism on $(U,\Delta)$, by Definition 5.3 and Corollary 4.2(2), we have $f(\cap\mathbf{P}) = f(\cap\Delta)$. Hence $f^{-1}(f(\cap\mathbf{P})) = f^{-1}(f(\cap\Delta))$. By Definition 5.3 and Corollary 4.1(2), $\cap\mathbf{P} = \cap\Delta$. Assume that $\exists\mathbf{C}_i \in \mathbf{P}$ such that $\cap\{\mathbf{P} - \{\mathbf{C}_i\}\} = \cap\Delta$, then $f(\cap\{\mathbf{P} - \{\mathbf{C}_i\}\}) = f(\cap\Delta)$. Again, by Definition 5.3 and Corollary 4.2(2), we have $\cap f(\mathbf{P} - \{\mathbf{C}_i\}) = \cap f(\Delta)$. Hence $\cap\{f(\mathbf{P}) - f(\mathbf{C}_i)\} = \cap f(\Delta)$. This is a contradiction to that $f(\mathbf{P})$ is a reduct of $f(\Delta)$. This completes the proof of this theorem. □

Parallel to Theorem 5.1, there is the following theorem for a homomorphism from $(V,\Gamma)$ to $(U,f^{-1}(\Gamma))$.

**Theorem 5.2.** *Suppose that $(V,\Gamma)$ is a covering information system, $(U,f^{-1}(\Gamma))$ is an induced covering information system of $(V,\Gamma)$, and $f^{-1}$ is a homomorphism from $(V,\Gamma)$ to $(U,f^{-1}(\Gamma))$ and $\mathbf{Q} \subseteq \Gamma$. Then $\mathbf{Q}$ is a reduct of $\Gamma$ if and only if $f^{-1}(\mathbf{Q})$ is a reduct of $f^{-1}(\Gamma)$.*

**Proof.** It is similar to the proof of Theorem 5.1. □

The following example is employed to illustrate that attribute reductions of the original system and image system are equivalent to each other under the condition of homomorphism. That means that we can simplify the method for attribute reductions of complex information systems by constructing a proper homomorphism in communication between different information systems.

**Example 5.1.** Let $(U,\Delta = \{\mathbf{C}_1,\mathbf{C}_2,\mathbf{C}_3,\mathbf{C}_4\})$ be a covering information system, where $U = \{x_1,x_2,\ldots,x_{20}\}$ and

$$\mathbf{C}_1 = \{\{x_2,x_6,x_7,x_8,x_{10},x_{15},x_{16},x_{18}\},\{x_4,x_6,x_8,x_9,x_{10},x_{13},x_{14},x_{18},x_{19}\},\{x_1,x_3,x_5,x_{11},x_{12},x_{17},x_{20}\}\}$$

$$\mathbf{C}_2 = \{\{x_1,x_2,x_6,x_7,x_8,x_{10},x_{11},x_{12},x_{15},x_{16},x_{18},x_{20}\},\{x_2,x_7,x_{15},x_{16}\},\{x_3,x_4,x_5,x_9,x_{13},x_{14},x_{17},x_{19}\}\}$$

$$\mathbf{C}_3 = \{\{x_1,x_{11},x_{12},x_{20}\},\{x_2,x_3,x_5,x_7,x_{15},x_{16},x_{17}\},\{x_3,x_4,x_5,x_9,x_{13},x_{14},x_{19}\},\{x_2,x_3,x_5,x_6,x_7,x_8,x_{10},x_{15},x_{16},x_{17},x_{18}\}\}$$

$$\mathbf{C}_4 = \{\{x_2,x_3,x_5,x_7,x_{15},x_{16},x_{17}\},\{x_3,x_4,x_5,x_9,x_{13},x_{14},x_{17},x_{19}\},..\{x_1,x_3,x_5,x_6,x_8,x_{10},x_{11},x_{12},x_{17},x_{18},x_{20}\}\}.$$

Thus $\Delta_{x_2} = \Delta_{x_7} = \Delta_{x_{15}} = \{x_2,x_7,x_{15},x_{16}\}$, $\Delta_{x_3} = \Delta_{x_5} = \{x_3,x_5,x_{17}\}$, $\Delta_{x_6} = \Delta_{x_8} = \Delta_{x_{10}} = \{x_6,x_8,x_{10},x_{18}\}$, $\Delta_{x_1} = \Delta_{x_{11}} = \Delta_{x_{12}} = \{x_1,x_{11},x_{12},x_{19}\}$, $\Delta_{x_4} = \Delta_{x_9} = \Delta_{x_{13}} = \Delta_{x_{14}} = \{x_4,x_9,x_{13},x_{14},x_{20}\}$. Let $V = \{y_1,y_2,y_3,y_4,y_5\}$, Define a mapping $f$: $U \to V$ as follows:

| $x_1,x_{11},x_{12},x_{20}$ | $x_2,x_7,x_{15},x_{16}$ | $x_3,x_5,x_{17}$ | $x_4,x_9,x_{13},x_{14},x_{19}$ | $x_6,x_8,x_{10},x_{18}$ |
|---|---|---|---|---|
| $y_1$ | $y_2$ | $y_3$ | $y_4$ | $y_5$ |

Then $f(\Delta) = \{f(\mathbf{C}_1),f(\mathbf{C}_2),f(\mathbf{C}_3)\}$, where $f(\mathbf{C}_1) = \{\{y_2,y_5\}, \{y_4,y_5\}, \{y_1,y_3\}\}$, $f(\mathbf{C}_2) = \{\{y_1,y_2,y_5\}, \{y_2\}, \{y_3,y_4\}\}$, $f(\mathbf{C}_3) = \{\{y_1\}, \{y_2,y_3\}, \{y_3,y_4\}, \{y_2,y_3,y_5\}\}$, $f(\mathbf{C}_4) = \{\{y_2,y_3\}, \{y_3,y_4\}, \{y_1,y_3,y_5\}\}$. Hence $(V,f(\Delta))$ is the $f$-induced covering information system of $(U,\Delta)$. It is very easy to verify that $f$ is a homomorphism on $(U,\Delta)$. We can see that $f(\mathbf{C}_3)$ and $f(\mathbf{C}_4)$ is superfluous in $f(\Delta)$ if and only if that $\mathbf{C}_3$ and $\mathbf{C}_4$ is superfluous in $\Delta$ and that $\{f(\mathbf{C}_1),f(\mathbf{C}_2)\}$ is a reduct of $f(\Delta)$ is equivalent to that $\{\mathbf{C}_1,\mathbf{C}_2\}$ is a reduct of $\Delta$. □

**Remark.** It is noteworthy that the covering information system $(U, \Delta)$ has great size while its image system $(V, f(\Delta))$ has the relatively small volume. This means that for a given great-size covering information system, we can find the relatively small image system that has the same reducts as the original database. Therefore, we can change a large-size database into a smaller one by adopting homomorphism technique. Based on the technique we can quickly perform equivalent attribute reductions and rule extraction in the smaller compressed image database and so make decisions.

## 6. Homomorphisms between covering decision systems and their properties

In this section, we investigate some invariant properties of communication between differrent covering decision systems under the condition of homomorphisms. Covering decision systems include consistent covering decision systems and inconsistent decision systems. Next, we start our study on consistent covering decision systems by defining some related notions first.

**Definition 6.1.** Let $\mathbf{\Delta} = \{\mathbf{C}_i : i = 1, \ldots, m\}$ be a family of coverings on $U$, $D$ a decision equivalence relation on $U$ and $U/D = \{[x]_D : x \in U\}$ the decision partition relative to $\Delta$. If $\Delta_x \subseteq [x]_D$ for any $x \in U$, then $(U, \mathbf{\Delta}, D)$ is called a consistent covering decision system, denoted as $Cov(\mathbf{\Delta}) \leqslant U/D$. Otherwise, $(U, \mathbf{\Delta}, D)$ is called an inconsistent covering decision system. The positive domain of $D$ relative to $\mathbf{\Delta}$ is defined as $Pos_{\Delta}(D) = \cup_{X \in U/D} \underline{apr}_{\mathbf{\Delta}} X$.

According to Definition 6.1, we can conclude that $(U, \mathbf{\Delta}, D)$ is an inconsistent covering decision system if and only if $POS_{\Delta}(D) \neq U$.

**Definition 6.2.** Let $(U, \mathbf{\Delta}, D)$ be a consistent covering decision system. For$\mathbf{C}_i \in \mathbf{\Delta}$, if $Cov(\mathbf{\Delta} - \{\mathbf{C}_i\}) \leqslant U/D$, then $\mathbf{C}_i$ is called superfluous relative to $D$ in $\mathbf{\Delta}$, otherwise $\mathbf{C}_i$ is called indispensable relative to $D$ in $\mathbf{\Delta}$. Let $\mathbf{P} \subseteq \mathbf{\Delta}$, if $Cov(\mathbf{P}) \leqslant U/D$ and every element in $\mathbf{P}$ is indispensable in $\mathbf{P}$, i.e., for every $\mathbf{C}_i \in \mathbf{P}$, $COV(\mathbf{P} - \{\mathbf{C}_i\}) \leqslant U/D$ is not true, then $\mathbf{P}$ is called a reduct of $\mathbf{\Delta}$ relative to $D$, relative reduct in short. The collection of all the indispensable elements in $\mathbf{\Delta}$ is called the core of $\mathbf{\Delta}$ relative to $D$, relative core in short, denoted as $Core_D(\mathbf{\Delta})$.

From Definition 6.2, we know that a relative reduct of a consistent covering decision system is the minimal subset of conditional coverings (attributes) to ensure every decision rule is still consistent.

**Definition 6.3.** Let $U$ and $V$ be two finite universes, $(U, \Delta, D)$ a covering decision system and $f: U \to V$ a surjection from $U$ to $V$. Then $(V, f(\Delta), f(D))$ is referred to as a $f$-induced covering decision system of $(U, \Delta, D)$. For any $X \in U/D$, if $f$ is a consistent function with respect to $X$, then $f$ is called consistent with respect to $D$.

In the following, we discuss some basic properties of a consistent covering decision system under some homomorphisms.

**Theorem 6.1.** *Let $(U, \Delta, D)$ be a covering decision system, $U/D = \{[x]_D : x \in U\}$ a decision partition on $U$ and $(V, f(\Delta), f(D))$ a $f$-induced covering decision system of $(U, \Delta, D)$. Then the following statements are equivalent:*

(1) *$f$ is consistent with respect to $D$;*
(2) *For any $x, y \in U$, if $[x]_D \cap [y]_D = \emptyset$, then $f([x]_D) \cap f([y]_D) = \emptyset$. That is, $\{f([x]_D) : x \in U\}$ is a partition on $V$, and then we denote $f([x]_D)$ by $[f(x)]_{f(D)}$ for any $x \in U$;*
(3) *$f^{-1}(f([x]_D)) = [x]_D$, $\forall x \in U$.*

**Proof**

(1) ⇒ (2) Assume that $\exists x_0, y_0 \in U$ satisfying $[x_0]_D \cap [y_0]_D = \emptyset$ and $f([x_0]_D) \cap f([y_0]_D) \neq \emptyset$. Let $u \in f([x_0]_D) \cap f([y_0]_D)$, then there exist $x_1 \in [x_0]_D$ and $y_1 \in [y_0]_D$ such that $f(x_1) = f(y_1) = u$. Since $f$ is consistent with respect to $D$, it follows from Definitions 3.1 and 6.3 that $x_1 \in [y_0]_D$ and $y_1 \in [x_0]_D$. This implies that $[x_0]_D \cap [y_0]_D \neq \emptyset$, a contradiction. Hence for any $x, y \in U$, if $[x]_D \cap [y]_D = \emptyset$, then $f([x]_D) \cap f([y]_D) = \emptyset$, i.e., $\{f([x]_D) : x \in U\}$ is a partition on $V$.
(2) ⇒ (1) Assume that $f$ is not consistent with respect to $D$, then there exists $X_0 \in U/D$ such that $f$ is not consistent with respect to $X_0$. By Definition 3.1, there exists $x_0, y_0 \in X_0$ satisfying $f(x_0) = f(y_0)$ such that $x_0 \in X_0$ and $y_0 \notin X_0$, which implies that $X_0 = [x_0]_D$ and $[x_0]_D \cap [y_0]_D = \emptyset$. This, together with the fact $f(x_0) = f(y_0)$, means that $f([x]_D) \cap f([y]_D) \neq \emptyset$, a contradiction. Therefore $f$ is consistent with respect to $D$.
(1) ⇒ (3) It is obvious that $f^{-1}(f([x]_D)) \supseteq [x]_D$. We only need to prove the inverse inclusion. For any $y \in f^{-1}(f([x]_D))$, we have $f(y) \in f([x]_D)$. Thus $\exists z \in [x]_D$ such that $f(y) = f(z)$. Since $f$ is consistent with respect to $D$, it follows from Definitions 3.1 and 6.3 that $y \in [x]_D$. Hence $f^{-1}(f([x]_D)) \subseteq [x]_D$. So we have $f^{-1}(f([x]_D)) = [x]_D$.
(3) ⇒ (1) For any $x, y \in U$ satisfying $f(x) = f(y)$, we only need to verify that $y \in [x]_D$ by Definitions 3.1 and 6.3. By $x \in [x]_D$ we have $f(y) \in f([x]_D)$. Thus $f^{-1}(f(y)) \subseteq f^{-1}(f([x]_D))$. This, together with $y \in f^{-1}(f(y))_f$, gives rise to that $y \in f^{-1}(f([x]_D))$. It follows from the fact that $f^{-1}(f([x]_D)) = [x]_D$ that $y \in [x]_D$, as desired. □

The following theorem delineates the relationship between homomorphism and decision consistence of a consistent covering decision system.

**Theorem 6.2.** *Let $(U,\Delta,D)$ be a consistent covering decision system. If f is a homomorphism from $(U,\Delta)$ to $(V,f(\Delta))$, then f is consistent with respect to D.*

**Proof.** In order to prove that $f$ is consistent with respect to $D$, it suffices to show that for any $x\in U$, $f$ is consistent with respect to $[x]_D$. Since $(U,\Delta,D)$ is a consistent covering decision system, we have that $\Delta_u\subseteq[u]_D=[x]_D$ for any $u\in[x]_D$. Since $f$ is a homomorphism on $(U,\Delta)$, it follows from Definitions 3.1 and 5.3 that $\Delta_u=\Delta_v$ for any $v\in V$ satisfying $f(v)=f(u)$, which implies that $\Delta_v\subseteq[x]_D$. This, together with the reflexivity of $\Delta_v$, implies that $v\in[x]_D$. By Definition 3.1, we know that $f$ is consistent with respect to $[x]_D$, as desired. □

The following theorem present a necessary and sufficient condition that attribute reductions in a consistent covering decision system and its image system are equivalent to each other under certain conditions.

**Theorem 6.3.** *Let $(U,\Delta,D)$ be a consistent covering decision system and $(V,f(\Delta),f(D))$ a f-induced covering decision system of $(U,\Delta,D)$. If f is a homomorphism from $(U,\Delta)$ to $(V,f(\Delta))$, then*

(1) *$(V,f(\Delta),f(D))$ is a consistent covering decision system.*
(2) *Let $\mathbf{P}\subseteq\Delta$, then $\mathbf{P}$ is a reduct of $\boldsymbol{\Delta}$ relative to D if and only if $f(\mathbf{P})$ is a reduct of $f(\Delta)$ relative to $f(D)$.*

**Proof**

(1) (1) Since $f$ is surjective, it follows that for any $u\in V$, there exists $x\in U$ such that $f(x)=u$. Since $(U,\Delta,D)$ is a consistent covering decision system, we have that $\Delta_x\subseteq[x]_D$. Thus $f(\Delta_x)\subseteq f([x]_D)$. Since $f$ is a homomorphism from $(U,\Delta)$ to $(V,f(\Delta))$, it follows from Definition 5.3 and Theorem 4.3(1) that $f(\Delta_x)=f(\Delta)_{f(x)}$, which implies $f(\Delta)_{f(x)}\subseteq f([x]_D)$. By Theorem 6.1(2) and Theorem 6.2, we have $f([x]_D)=[f(x)]_{f(D)}$. Thus $f(\Delta)_{f(x)}\subseteq[f(x)]_{f(D)}$, i.e., $f(\Delta)_u\subseteq[u]_{f(D)}$. Hence $(V,f(\Delta),f(D))$ is a consistent covering decision system.

(2) (2) $\Rightarrow$ Since $\mathbf{P}$ is a reduct of $\Delta$, we have that $\mathbf{P}_x\subseteq[x]_D$ for any $x\in U$. Thus $f(\mathbf{P}_x)\subseteq f([x]_D)$. Since $f$ is a homomorphism on $(U,\Delta)$, it follows from Theorem 4.3(1), Theorem 6.1(2) and Theorem 6.2 that $f(\mathbf{P})_{f(x)}=f(\mathbf{P}_x)\subseteq f([x]_D)=[f(x)]_{f(D)}$. Assume that $\exists f(\mathbf{C}_i)\in f(\mathbf{P})$ such that $f(\mathbf{P}-\{\mathbf{C}_i\})_{f(x)}\subseteq f([x]_D)$. By Theorem 4.4(1) we know that $f((\mathbf{P}-\{\mathbf{C}_i\})_x)=f(\mathbf{P}-\{\mathbf{C}_i\})_{f(x)}\subseteq f([x]_D)$. $\{\mathbf{C}_i\})_x)=f(\mathbf{P}-\{\mathbf{C}_i\})_{f(x)}\subseteq f([x]_D)$. Since $(U,\Delta,D)$ is a consistent covering decision system and $f$ is a homomorphism on $(U,\Delta)$, it follows from Theorem 6.2, Corollary 3.3 and Theorem 6.1(3), $(\mathbf{P}-\{\mathbf{C}_i\})_x=f^{-1}(f((\mathbf{P}-\{\mathbf{C}_i\})_x))\subseteq f^{-1}(f([x]_D))=[x]_D$. $=f^{-1}(f((\mathbf{P}-\{\mathbf{C}_i\})_x))\subseteq f^{-1}(f([x]_D))=[x]_D$. This is a contradiction to that $\mathbf{P}$ is a reduct of $\Delta$. Hence $f(\mathbf{P})$ is a relative reduct of $f(\Delta)$.
$\Leftarrow$ Let $f(\mathbf{P})$ is a reduct of $f(\Delta)$ relative to $f(D)$, then $f(\mathbf{P})_{f(x)}\subseteq[f(x)]_{f(D)}$ for any $x\in U$. Since $f$ is a homomorphism on $(U,\Delta)$, it follows from Theorem 4.3(1), Theorem 6.1(2) and Theorem 6.2 that $f(\mathbf{P}_x)=f(\mathbf{P})_{f(x)}\subseteq[f(x)]_{f(D)}=f([x]_D)$. Thus $f^{-1}(f(\mathbf{P}_x))\subseteq f^{-1}(f([x]_D))$. By Corollary 3.3 and Theorem 6.1(3), we have $\mathbf{P}_x\subseteq[x]_D$. Assume that there exists $\mathbf{C}_i\in\Delta$ such that $(\mathbf{P}-\{\mathbf{C}_i\})_x\subseteq[x]_D$, then $f((\mathbf{P}-\{\mathbf{C}_i\})_x)\subseteq f([x]_D)$. Similarly, by Theorem 4.3(1), Theorem 6.1(2) and Theorem 6.2, we have that $(f(\mathbf{P})-f(\mathbf{C}_i))_{f(x)}=f((\mathbf{P}-\{\mathbf{C}_i\})_x)\subseteq f([x]_D)=[f(x)]_{f(D)}$. This is a contradiction to that $f(\mathbf{P})$ is a reduct of $f(\Delta)$ relative to $f(D)$. Hence $\mathbf{P}$ is a reduct of $\boldsymbol{\Delta}$ relative to $D$. □

By Theorem 6.3 we immediately get the following corollary.

**Corollary 6.1.** *Let $(U,\Delta,D)$ be a consistent covering decision system, $(V,f(\Delta),f(D))$ a f-induced covering decision system of $(U,\Delta,D)$, f a homomorphism on $(U,\Delta)$, $\mathbf{C}_i\in\Delta$ and $\mathbf{P}\subseteq\Delta$. Then*

(1) *$\mathbf{C}_i$ is indispensable with respect to D in $\Delta$ if and only if $f(\mathbf{C}_i)$ is indispensable in $f(\Delta)$ with respect to $f(D)$;*
(2) *$\mathbf{P}$ is superfluous with respect to D in $\Delta$ if and only if $f(\mathbf{P})$ is superfluous in $f(\Delta)$ with respect to $f(D)$;*
(3) *The image of the relative core of $\Delta$ is the relative core of the image of $\Delta$ and the inverse image of the relative core of $f(\Delta)$ is the relative core of $\Delta$. That is, $f(Core_D(\Delta))=Core_{f(D)}(f(\Delta))$ and $f^{-1}(Core_{f(D)}(f(\Delta)))=Core_D(\Delta)$.*

**Example 6.2.** Let $(U,\Delta=\{\mathbf{C}_1,\mathbf{C}_2,\mathbf{C}_3,\mathbf{C}_4\},D)$ be a consistent covering decision system, where $U=\{x_1,x_2,\ldots,x_{15}\}$,

$$\mathbf{C}_1=\{\{x_1,x_2,x_3,x_4,x_5,x_8,x_{10},x_{15}\},\{x_3,x_5,x_7,x_{11},x_{12}\},\{x_4,x_6,x_8,x_9,x_{10},x_{13},x_{14}\}\};$$

$$\mathbf{C}_2=\{\{x_1,x_2,x_3,x_4,x_5,x_7,x_8,x_{10},x_{11},x_{12},x_{15}\},\{x_3,x_4,x_5,x_8,x_{10}\},\{x_3,x_5,x_6,x_9,x_{13},x_{14}\}\};$$

$$\mathbf{C}_3=\{\{x_1,x_2,x_3,x_4,x_5,x_8,x_{10},x_{15}\},\{x_6,x_9,x_{13},x_{14}\},\{x_4,x_7,x_8,x_{10},x_{11},x_{12}\}\};$$

$$\mathbf{C}_4 = \{\{x_1, x_2, x_3, x_5, x_{15}\}, \{x_6, x_7, x_9, x_{11}, x_{12}, x_{13}, x_{14}\}, \{x_3, x_4, x_5, x_8, x_{10}\}\}.$$

$$U/D = \{X_1, X_2, X_3\}, X_1 = \{x_1, x_2, x_3, x_5, x_{15}\}, \quad X_2 = \{x_4, x_7, x_8, x_{10}, x_{11}, x_{12}\}, X_3 = \{x_6, x_9, x_{13}, x_{14}\}.$$

$$\Delta_{x_3} = \Delta_{x_5} = \{x_3, x_5\}, \quad \Delta_{x_1} = \Delta_{x_2} = \Delta_{x_{15}} = \{x_1, x_2, x_3, x_5, x_{15}\}, \quad \Delta_{x_4} = \Delta_{x_8} = \Delta_{x_{10}} = \{x_4, x_8, x_{10}\},$$

$$\Delta_{x_6} = \Delta_{x_9} = \Delta_{x_{13}} = \Delta_{x_{14}} = \{x_6, x_9, x_{13}, x_{14}\}, \quad \Delta_{x_7} = \Delta_{x_{11}} = \Delta_{x_{12}} = \{x_7, x_{11}, x_{12}\}.$$

Let $V = \{y_1, y_2, y_3, y_4, y_5\}$. Define a mapping $f$: $U \rightarrow V$ as follows:

| $x_1, x_2, x_{15}$ | $x_3, x_5$ | $x_4, x_8, x_{10}$ | $x_6, x_9, x_{13}, x_{14}$ | $x_7, x_{11}, x_{12}$ |
|---|---|---|---|---|
| $y_1$ | $y_2$ | $y_3$ | $y_4$ | $y_5$ |

Then $f(\mathbf{C}_1) = \{\{y_1, y_2, y_3\}, \{y_3, y_4\}, \{y_2, y_5\}\}$, $f(\mathbf{C}_2) = \{\{y_1, y_2, y_3, y_5\}, \{y_2, y_3\}, \{y_2, y_4\}\}$, $f(\mathbf{C}_3) = \{\{y_1, y_2, y_3\}, \{y_3, y_5\}, \{y_4\}\}$, $f(\mathbf{C}_4) = \{\{y_1, y_2\}, \{y_2, y_3\}, \{y_4, y_5\}\}$, $f(\Delta) = \{f(\mathbf{C}_1), f(\mathbf{C}_2), f(\mathbf{C}_3)\}$, and $V/f(D) = \{f(X_1), f(X_2), f(X_3)\}$, where $f(X_1) = \{y_1, y_2\}$, $f(X_2) = \{y_3, y_5\}$, $f(X_3) = \{y_4\}$. Then $(V, f(\Delta), f(D))$ is a $f$-induced covering decision system of $(U, \Delta, D)$. It is easy to verify that $f$ is a homomorphism from $(U, \Delta)$ to $(V, f(\Delta))$ and that $(V, f(\Delta), f(D))$ is a consistent covering decision system. □

**Remark.** We can observe that the data scale in the image system $(V, f(\Delta), f(D))$ is much smaller than the data scale in the original system. From Theorem 6.3, we can conclude that the attribute reductions of the original system and image system are equivalent to each other under the condition of homomorphism. Therefore, we can reduce the original system by reducing its image system.

Next, we begin to study some invariant properties of communication between differrent inconsistent covering decision systems. Let us start with introducing the relevant concepts of attribute reductions of an inconsistent covering decision system.

**Definition 6.4.** Let $(U, \mathbf{\Delta}, D)$ be an inconsistent covering decision system and $\mathbf{C}_i \in \Delta$. If $Pos_{\Delta}(D) = Pos_{\Delta - \{\mathbf{C}_i\}}(D)$, then $\mathbf{C}_i$ is called superfluous relative to $D$ in $\mathbf{\Delta}$, otherwise $\mathbf{C}_i$ is called indispensable relative to $D$ in $\mathbf{\Delta}$. Let $\mathbf{P} \subseteq \mathbf{\Delta}$, if $Pos_{\mathbf{P}}(D) = Pos_{\Delta}(D)$ and each element in $\mathbf{P}$ is indispensable relative to $D$ in $\mathbf{P}$, i.e., $Pos_{\mathbf{P}}(D) \neq Pos_{\mathbf{P} - \{\mathbf{C}_i\}}(D)$ for every $\mathbf{C}_i \in \mathbf{P}$, then $\mathbf{P}$ is called a reduct of $\mathbf{\Delta}$ relative to $D$, relative reduct in short. The collection of all the indispensable elements in $\mathbf{\Delta}$ is called the core of $\mathbf{\Delta}$ relative to $D$, denoted as $Core_D(\mathbf{\Delta})$.

For inconsistent covering decision systems, we have the same results on attribute reductions as consistent covering decision systems. That is, the attribute reductions of the original system and image system are also equivalent to each other under the condition of homomorphism. To prove this, it is convenient to have the following lemmas.

**Lemma 6.4.** *Let $(U, \mathbf{\Delta}, D)$ be an inconsistent covering decision system and $\mathbf{P} \subseteq \mathbf{\Delta}$. Then $Pos_{\mathbf{P}}(D) = Pos_{\Delta}(D)$ if only and if $\underline{apr}_{\mathbf{P}}(X) = \underline{apr}_{\mathbf{\Delta}}(X)$ for any $X \in U/D$.*

**Proof.** Since $\mathbf{P} \subseteq \mathbf{\Delta}$, then we have $\Delta_{x_i} \subseteq \mathbf{P}_{x_i}$ for any $x_i \in U$. By the definitions of $\underline{apr}_{\mathbf{\Delta}}(X)$ and $\underline{apr}_{\mathbf{P}}(X)$, it follows that $\underline{apr}_{\mathbf{P}}(X) \subseteq \underline{apr}_{\mathbf{\Delta}}(X)$ for any $X \in U/D$. If $\exists X_0 \in U/D$ such that $\underline{apr}_{\mathbf{P}}(X_0) \subset \underline{apr}_{\mathbf{\Delta}}(X_0)$, then $Pos_{\mathbf{P}}(D) \subset Pos_{\Delta}(D)$. This is a contradiction. Conversely, it is obviously true. □

**Lemma 6.5.** *Let $(U, \mathbf{\Delta}, D)$ be an inconsistent covering decision system and $(V, f(\Delta), f(D))$ a $f$-induced covering decision system of $(U, \Delta, D)$. If $f$ is a homomorphism from $(U, \Delta)$ to $(V, f(\Delta))$ and $f$ is consistent with respect to $D$, then $\underline{apr}_{f(\Delta)} f(X) = f(\underline{apr}_{\Delta} X) = \underline{apr}_{f(\Delta)} \underline{apr}_{f(\Delta)} f(X)$ for any $X \in U/D$.*

**Proof.** First, we prove that $\underline{apr}_{\Delta} X = \underline{apr}_{\Delta}(\underline{apr}_{\Delta} X)$ for any $X \subseteq U$. Since it is obvious that $\underline{apr}_{\Delta}(\underline{apr}_{\Delta} X) \subseteq \underline{apr}_{\Delta} X$, we only need to prove the inverse inclusion. For any $y \in \underline{apr}_{\Delta} X$, we have that $\Delta_y \subseteq X$. By the transitivity of $\Delta_y$, we have $\Delta_z \subseteq \Delta_y$ for any $z \in \Delta_y$. Thus $\Delta_z \subseteq X$, which implies $z \in \underline{apr}_{\Delta} X$. Hence $\Delta_y \subseteq \underline{apr}_{\Delta} X$. This means that $y \in \underline{apr}_{\Delta}(\underline{apr}_{\Delta} X)$ by the reflexivity of $\Delta_y$. Hence $\underline{apr}_{\Delta} X \subseteq \underline{apr}_{\Delta}(\underline{apr}_{\Delta} X)$.

Next, we prove that $\underline{apr}_{f(\Delta)} f(X) = \underline{apr}_{f(\Delta)} f(\underline{apr}_{\Delta} X)$. Since it is obvious that $\underline{apr}_{f(\Delta)} f(\underline{apr}_{\Delta} X) \subseteq \underline{apr}_{f(\Delta)} f(X)$, we only need to prove the inverse inclusion. For any $u \in \underline{apr}_{f(\Delta)} f(X)$, we have $f(\Delta)_u \subseteq f(X)$. Let $x \in U$ be such that $f(x) = u$, then $f(\Delta)_{f(x)} \subseteq f(X)$. Since $f$ is a homomorphism on $(U, \Delta)$, it follows that from Definition 5.3 and Theorem 4.3(1) that $f(\Delta_x) = f(\Delta)_{f(x)}$. Thus $f(\Delta_x) \subseteq f(X)$. Hence $f^{-1}(f(\Delta_x)) \subseteq f^{-1}(f(X))$. Again, since $f$ is a homomorphism on $(U, \Delta)$, by Corollary 3.3 we have $f^{-1}(f(\Delta_x)) = \Delta_x$. Since $f$ is consistent with respect to $D$, it follows from Theorem 6.1(3) that $f^{-1}(f(X)) = X$. Thus $\Delta_x \subseteq X$, which implies $x \in \underline{apr}_{\Delta} X$. Hence $u = f(x) \in f(\underline{apr}_{\Delta} X)$. This, together with the fact $\underline{apr}_{\Delta} X = \underline{apr}_{\Delta}(\underline{apr}_{\Delta} X)$ that $\underline{apr}_{f(\Delta)} f(X) \subseteq f(\underline{apr}_{\Delta} X) = f(\underline{apr}_{\Delta}(\underline{apr}_{\Delta} X))$. Since $f$ is a homomorphism from $(U, \Delta)$ to $(V, f(\Delta))$ and $\underline{apr}_{\Delta} X$ is $\Delta$-definable subset, by Theorem 4.8(2) we have $f(\underline{apr}_{\Delta}(\underline{apr}_{\Delta} X)) = \underline{apr}_{f(\Delta)} f(\underline{apr}_{\Delta} X)$. Thus $\underline{apr}_{f(\Delta)} f(X) \subseteq f(\underline{apr}_{\Delta} X) = \underline{apr}_{f(\Delta)} f(\underline{apr}_{\Delta} X)$. Hence $\underline{apr}_{f(\Delta)} f(X) = f(\underline{apr}_{\Delta} X) = \underline{apr}_{f(\Delta)} f(\underline{apr}_{\Delta} X)$. This completes the proof of this theorem. □

The following theorem present a necessary and sufficient condition that attribute reductions in an inconsistent covering decision system and its image system are equivalent to each other under certain conditions.

**Theorem 6.6.** *Let $(U,\boldsymbol{\Delta},D)$ be an inconsistent covering decision system and $(V,f(\Delta),f(D))$ a f-induced covering decision system of $(U,\Delta,D)$. If f is a homomorphism from $(U,\Delta)$ to $(V,f(\Delta))$ and f is consistent with respect to D, then*

(1) *$(V,f(\Delta),f(D))$ is an inconsistent covering decision system;*
(2) *Let $\mathbf{P}\subseteq\Delta$, then $\mathbf{P}$ is a reduct of $\boldsymbol{\Delta}$ relative to D if and only if $f(\mathbf{P})$ is a reduct of $f(\Delta)$ relative to $f(D)$.*

**Proof**

(1) Since $(U,\boldsymbol{\Delta},D)$ is an inconsistent covering decision system, it follows that $\exists x\in U$ such that $\Delta_x\subseteq[x]_D$ is not true. Thus there exists $y\in U$ satisfying $[x]_D\cap[y]_D=\emptyset$ and $\Delta_x\cap[y]_D\neq\emptyset$, which implies that there exist $u,v\in\Delta_x$ such that $u\in[x]_D$ and $v\in[y]_D$. Thus $f(u)\in f([x]_D)$ and $f(v)\in f([y]_D)$. Hence $f(\Delta_x)\cap f([x]_D)\neq\emptyset$ and $f(\Delta_x)\cap f([y]_D)\neq\emptyset$. Since $f$ is consistent with respect to $D$, it follows from Theorem 6.1(2) that $f([x]_D)\cap f([y]_D)=\emptyset$ and $f([x]_D)=[f(x)]_{f(D)}$. Hence $f(\Delta_x)\subseteq f([x]_D)$ is not true. Since $f$ is a homomorphism from $(U,\Delta)$ to $(V,f(\Delta))$, by Definition 5.3 and Theorem 4.4(1) we have that $f(\Delta_x)=f(\Delta)_{f(x)}$. Hence $f(\Delta)_{f(x)}\subseteq[f(x)]_{f(D)}$ is not true. Let $f(x)=w\in V$, then $f(\Delta)_w=f(\Delta)_{f(x)}$. Thus $f(\Delta)_w\subseteq[w]_{f(D)}$ is not true. Hence $(V,f(\Delta),f(D))$ is an inconsistent covering decision system.
(2) $\Rightarrow$ Since $f$ is a homomorphism from $(U,\Delta)$ to $(V,f(\Delta))$ and $f$ is consistent with respect to $D$, it follows from Lemma 6.5 that

$$f(Pos_{\Delta}(D))=f\left(\bigcup_{X\in U/D}\underline{apr}_{\Delta}X\right)=\bigcup_{X\in U/D}f(\underline{apr}_{\Delta}X)=\bigcup_{X\in U/D}\underline{apr}_{f(\Delta)}f(X)=Pos_{f(\Delta)}f(D).$$

Similarly, for any $\mathbf{P}\subseteq\Delta$, we have that

$$f(Pos_{\mathbf{P}}(D))=f\left(\bigcup_{X\in U/D}\underline{apr}_{\mathbf{P}}X\right)=\bigcup_{X\in U/D}f(\underline{apr}_{\mathbf{P}}X)=\bigcup_{X\in U/D}\underline{apr}_{f(\mathbf{P})}f(X)=Pos_{f(\mathbf{P})}f(D).$$

Since $\mathbf{P}$ is a reduct of $\boldsymbol{\Delta}$ relative to $D$, we have that $Pos_{\Delta}(D)=Pos_{\mathbf{P}}(D)$. Thus $f(Pos_{\Delta}(D))=f(Pos_{\mathbf{P}}(D))$. Hence $\bigcup_{X\in U/D}\underline{apr}_{f(\Delta)}f(X)=\bigcup_{X\in U/D}\underline{apr}_{f(\mathbf{P})}f(X)$. Assume that there exists $\mathbf{C}_i\in\mathbf{P}$ such that $Pos_{f(\mathbf{P})}f(D)=Pos_{f(\mathbf{P}-\{\mathbf{C}_i\})}f(D)$. Then $Pos_{f(\Delta)}f(D)=Pos_{f(\mathbf{P}-\{\mathbf{C}_i\})}f(D)$, that is, $\bigcup_{X\in U/D}\underline{apr}_{f(\Delta)}f(X)=\bigcup_{X\in U/D}\underline{apr}_{f(\mathbf{P}-\{\mathbf{C}_i\})}f(X)$. By Lemma 6.4 we have $\underline{apr}_{f(\Delta)}f(X)=\underline{apr}_{f(\mathbf{P}-\{\mathbf{C}_i\})}f(X)$ for any $X\in U/D$. By Lemma 6.5 we know that $f(\underline{apr}_{\Delta}X)=\underline{apr}_{f(\Delta)}f(X)=\underline{apr}_{f(\mathbf{P}-\{\mathbf{C}_i\})}f(X)=f(\underline{apr}_{\mathbf{P}-\{\mathbf{C}_i\}}X)$. Since $\underline{apr}_{\Delta}X$ and $\underline{apr}_{\mathbf{P}-\{\mathbf{C}_i\}}X$ are definable subsets with respect to $\Delta$ and $\mathbf{P}-\{\mathbf{C}_i\}$ respectively and $f$ is a homomorphism from $(U,\Delta)$ to $(V,f(\Delta))$, it follows that from Corollary 3.3 that $\underline{apr}_{\Delta}X=f^{-1}(f(\underline{apr}_{\Delta}X))=f^{-1}(f(\underline{apr}_{\mathbf{P}-\{\mathbf{C}_i\}}X))=\underline{apr}_{\mathbf{P}-\{\mathbf{C}_i\}}X$. Hence $Pos_{\Delta}(D)=Pos_{\mathbf{P}-\{\mathbf{C}_i\}}(D)$. This is a contradiction to that $\mathbf{P}$ is a reduct of $\boldsymbol{\Delta}$ relative to $D$.

$\Leftarrow$ Let $f(\mathbf{P})$ is a reduct of $f(\Delta)$ relative to $f(D)$, then $POS_{f(\mathbf{P})}f(D)=POS_{f(\Delta)}f(D)$, that is, $\bigcup_{X\in U/D}\underline{apr}_{f(\mathbf{P})}f(X)=\bigcup_{X\in U/D}\underline{apr}_{f(\Delta)}f(X)$. By Lemma 6.4, we have that $\underline{apr}_{f(\Delta)}f(X)=\underline{apr}_{f(\mathbf{P})}f(X)$ for any $X\in U/D$. By Theorem 4.8(5), it follows that $\underline{apr}_{f^{-1}(f(\Delta))}f^{-1}(f(X))=\underline{apr}_{f^{-1}(f(\mathbf{P}))}f^{-1}(f(X))$. Since $f$ is a homomorphism on $(U,\Delta)$ and $f$ is consistent with respect to $D$, it follows from Theorem 4.2(2) and 6.1(3) that $\underline{apr}_{\Delta}X=\underline{apr}_{\mathbf{P}}X$. Assume that there exists $\mathbf{C}_i\in\mathbf{P}$ such that $\underline{apr}_{\Delta}X=\underline{apr}_{\mathbf{P}-\{\mathbf{C}_i\}}X$, then $f(\underline{apr}_{\Delta}X)=f(\underline{apr}_{\mathbf{P}-\{\mathbf{C}_i\}}X)$. By Lemma 6.5, $\underline{apr}_{f(\Delta)}f(X)=f(\underline{apr}_{\Delta}X)=f(\underline{apr}_{\mathbf{P}-\{\mathbf{C}_i\}}X)=\underline{apr}_{f(\mathbf{P}-\{\mathbf{C}_i\})}f(X)$. Hence

$$POS_{f(\Delta)}f(D)=\bigcup_{X\in U/D}\underline{apr}_{f(\Delta)}f(X)=\bigcup_{X\in U/D}\underline{apr}_{f(\mathbf{P}-\{\mathbf{C}_i\})}f(X)=POS_{f(\mathbf{P}-\{\mathbf{C}_i\})}f(D).$$

This is a contradiction to that $f(\mathbf{P})$ is a reduct of $f(\Delta)$ relative to $f(D)$. □

By Theorem 6.6 we can quickly get the following corollary.

**Corollary 6.2.** *Suppose that $(V,f(\Delta),f(D))$ is a f-induced covering decision system of $(U,\Delta,D)$, f is a homomorphism from $(U,\Delta)$ to $(V,f(\Delta))$ and consistent with respect to D, $\mathbf{C}_i\in\Delta$ and $\mathbf{P}\subseteq\Delta$. Then*

(1) *$\mathbf{C}_i$ is indispensable with respect to D in $\Delta$ if and only if $f(\mathbf{C}_i)$ is indispensable in $f(\Delta)$ with respect to $f(D)$;*
(2) *$\mathbf{P}$ is superfluous with respect to D in $\Delta$ if and only if $f(\mathbf{P})$ is superfluous in $f(\Delta)$ with respect to $f(D)$;*
(3) *The image of the relative core of $\Delta$ is the relative core of the image of $\Delta$ and the inverse image of the relative core of $f(\mathbf{R})$ is the relative core of the original image. i.e., $f(Core_D(\Delta))=Core_{f(D)}(f(\Delta))$ and $f^{-1}(Core_{f(D)}(f(\Delta)))=Core_D(\Delta)$.*

*To better illustrate the potential applications of these properties into attribute reductions of covering decision systems, let us see the following example.*

**Example 6.3.** Let $(U,\Delta = \{\mathbf{C}_1,\mathbf{C}_2,\mathbf{C}_3,\mathbf{C}_4\},D)$ be an inconsistent covering decision system, where $U = \{x_1,x_2,\ldots,x_{15}\}$,

$$\mathbf{C}_1 = \{\{x_4,x_7,x_8,x_{10},x_{11},x_{15}\},\{x_3,x_4,x_5,x_8,x_{10},x_{13},x_{14}\},\{x_1,x_2,x_6,x_9,x_{12}\}\};$$

$$\mathbf{C}_2 = \{\{x_1,x_2,x_4,x_7,x_8,x_{10},x_{11},x_{12},x_{15}\},\{x_4,x_6,x_8,x_9,x_{10}\},\{x_3,x_5,x_6,x_9,x_{13},x_{14}\}\};$$

$$\mathbf{C}_3 = \{\{x_1,x_2,x_3,x_4,x_5,x_6,x_8,x_9,x_{10},x_{12},x_{13},x_{14}\},\{x_4,x_6,x_7,x_8,x_9,x_{10},x_{11},x_{15}\}\};$$

$$\mathbf{C}_4 = \{\{x_6,x_7,x_9,x_{11},x_{15}\},\{x_3,x_4,x_5,x_6,x_8,x_9,x_{10},x_{13},x_{14}\},\{x_1,x_2,x_3,x_5,x_{12},x_{13},x_{14}\}\}.$$

$$U/D = \{X_1,X_2,X_3\},\quad X_1 = \{x_1,x_2,x_3,x_5,x_{12}\},\quad X_2 = \{x_4,x_7,x_8,x_{10},x_{11},x_{15}\},\quad X_3 = \{x_6,x_9,x_{13},x_{14}\}.$$

Thus $\Delta_{x_1} = \Delta_{x_2} = \Delta_{x_{12}} = \{x_1,x_2,x_{12}\},\ \Delta_{x_4} = \Delta_{x_8} = \Delta_{x_{10}} = \{x_4,x_8,x_{10}\},\ \Delta_{x_3} = \Delta_{x_5} = \Delta_{x_{13}} = \Delta_{x_{14}} = \{x_3,x_5,x_{13},x_{14}\},\ \Delta_{x_6} = \Delta_{x_9} = \{x_6,x_9\},\ \Delta_{x_7} = \Delta_{x_{11}} = \Delta_{x_{15}} = \{x_7,x_{11},x_{15}\}$. Let $V = \{y_1,y_2,y_3,y_4,y_5\}$. Define a mapping $f$: $U \to V$ as follows:

$$\frac{x_1,x_2,x_{12}\quad x_3,x_5\quad x_4,x_8,x_{10}\quad x_6,x_9\quad x_7,x_{11},x_{15}\quad x_{13},x_{14}}{y_1\qquad y_2\qquad y_3\qquad y_4\qquad y_5\qquad y_6}.$$

Then $f(\Delta) = \{f(\mathbf{C}_1),f(\mathbf{C}_2),f(\mathbf{C}_3)\}$, where $f(\mathbf{C}_1) = \{\{y_3,y_5\},\ \{y_2,y_3,y_6\},\ \{y_1,y_4\}\}$, $f(\mathbf{C}_2) = \{\{y_1,y_3,y_5\},\ \{y_3,y_4\},\ \{y_2,y_4,y_6\}\}$. $f(\mathbf{C}_3) = \{\{y_3,y_4,y_5\},\ \{y_1,y_2,x_3,y_4,y_6\}\}$, $f(\mathbf{C}_4) = \{\{y_4,y_5\},\ \{y_2,y_3,y_4,y_6\},\ \{y_1,y_2,y_6\}\}$. $V/f(D) = \{f(X_1),f(X_2),f(X_3)\}$, where $f(X_1) = \{y_1,y_2\}$, $f(X_2) = \{y_3,y_5\}$, $f(X_3) = \{y_4,y_6\}$. Then $(V,f(\Delta),f(D))$ is a $f$-induced covering decision system of $(U,\Delta,D)$. It is easy to verify that $f$ is a homomorphism on $(U,\Delta)$ and consistent with respect to $D$, and that $(V,f(\Delta),f(D))$ is an inconsistent covering decision system. We can observe that the data scale in the image system $(V,f(\Delta),f(D))$ is much smaller than the data scale in the original system. From Theorem 6.6, we can conclude that the attribute reductions of the original system and image system are equivalent to each other under the condition of homomorphism. Therefore, we can reduce the original system by reducing the image system.

## 7. Conclusions

In this paper, we study the communications between covering information systems and between covering decision systems. We show that a mapping between two universes can induce a covering on one universe according to a given covering on the other universe. A covering information system or a covering decision system can be regarded as a combination of some covering approximation spaces on the same universe. A mapping between covering approximation spaces can be seen as a mapping between covering information systems. A homomorphism between covering information systems is a special covering mapping between covering information systems. Based on these observations, we have explored some invariant properties of covering mappings and homomorphisms in covering information systems and decision systems, and prove that attribute reductions of an original information system or decision system and its image system are equivalent to each other. These results show that some characters of a system can be guaranteed in image system. According to these statements, the concepts of homomorphisms between information systems can be viewed as a useful tool to study relationships between covering information systems. All the results may have useful applications in quick attribute reduction, decision making and reasoning about data, especially for the case of communication between two covering information systems.

In the future, we will apply the results to attribute reductions and feature selections for covering information systems. These topics will help form a systematic framework for theoretic analysis and practical applications of information communications.

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