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Fuzzy information systems and their homomorphisms

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Abstract

With the arrival of the information age, information acquisition and communication have become more and more important in the field of information technology. This paper uses the concept of homomorphism as a basic tool to study the communication between fuzzy information systems. The concepts of consistent and compatible mappings with respect to fuzzy sets are firstly defined and their basic properties are studied. Then, a pair of lower and upper rough fuzzy approximation operators is constructed by means of the concept of fuzzy mappings. Basic invariant properties of the approximation operators are investigated. Finally, the concepts of fuzzy information system and its homomorphism are introduced, and some invariant properties of fuzzy information systems under homomorphisms are examined. It is proved that the attribute reductions of an original information system and its image system are equivalent to each other in the context of fuzzy attributes. These results may have potential applications in attribute reduction and classification issues.

Keywords: Fuzzy mapping; Rough set; Homomorphism; Attribute reduction

1. Introduction

Information system is a database that describes the relationships between objects and attributes; it is one of the most important mathematical models in the field of artificial intelligence. With the arrival of the information age, access to information, analysis and synthesis of information have become a focus in the field of information technology. Information uncertainty analysis, information fusion, attribute reduction and knowledge classification are becoming more and more important.

Rough set theory [1–3] is a useful tool for dealing with uncertainty, vagueness, and incompleteness of information. It has been applied to rule extraction [4–9], reasoning with uncertainty [10–14], uncertainty modeling [15–22], classification and feature selection [23–29]. In recent years, the communication between information systems has been an important problem in granular computing [30–33]. Many topics on this problem have been widely investigated by

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using fuzzy sets and rough sets [34–41]. One of the main objectives of research on communication between information systems is to reduce large-stored data volume and redundant attributes while preserving some basic information [36–41].

Communication is directly related to the issue of transformations of information systems while preserving their basic properties. As explained in [31–33], communication allows one to translate the information contained in one granular world into the granularity of another granular world and thus provides a mechanism for exchanging information with other granular worlds. From the mathematical viewpoint, this kind of communication can be explained as comparing some structures and properties of different information systems via mappings, which are useful tools to study the relationship between information systems. Thus, the approximations and reductions in an original information system can be regarded as encoding, while the image system is seen as an interpretive information system. The notion of homomorphism based on rough set theory, introduced by Grzymala-Busse [34,35], can be viewed as a special mapping between information systems and used for aggregating sets of objects, attributes, and descriptors of the original system [35–41]. This idea perhaps is a good approach for reducing large-stored data volume and data attributes. In [35], Grzymala-Busse depicted the conditions which make an information system endomorphic. In [36], Deyu Li discussed some invariant properties of discrete information systems under homomorphism. In [37–40], Wang et al. investigated some invariant properties of generalized information systems under homomorphisms and proved that attribute reductions in the original system and image system are equivalent to each other under the condition of homomorphism.

In reality, there are a lot of information systems in which knowledge attributes are vague or ambiguous [42–47]. If each attribute induces a fuzzy set on the sample space, we call this kind of information system fuzzy information system. If all fuzzy sets induced by attributes can define a fuzzy covering on the sample space, the fuzzy information system is called fuzzy covering information system. Fuzzy information systems are extensively used in the fields of fuzzy decision making [42,45,47] and fuzzy comprehensive evaluations [43,44,46]. Since the topic on homomorphism between information systems has become an important problem, how to search for homomorphisms between fuzzy information systems and examine their invariant properties are our new issues.

In this paper, we introduce rough sets as a basic tool to study homomorphisms between fuzzy information systems. By Zadeh's extension principle [48], we define a class of special fuzzy mappings: consistent and compatible mappings and study their basic invariant properties. We then construct a pair of rough fuzzy approximation operators by means of fuzzy mappings and examine their main properties. Finally, we propose the concepts of homomorphism between fuzzy information systems and prove that the attribute reductions of a fuzzy information system and its image system are equivalent to each other under the condition of homomorphism.

The remainder of this paper is organized as follows. In Section 2, we review the relevant concepts in rough sets and fuzzy sets. In Section 3, we study some invariant properties of consistent and compatible mappings and construct a pair of rough fuzzy approximation operators by means of fuzzy mappings. In Section 4, we introduce the concept of homomorphism between fuzzy information systems and we get its main results. Conclusions are drawn in Section 5.

2. Some basic notions of fuzzy and rough sets

We first review some basic concepts related to classical rough sets that can be found in [1–3].

An information system is a pair $IS = (U, C)$, where U is a nonempty finite set of objects and $C = \{a_1, a_2, \dots, a_m\}$ is a nonempty and finite set of attributes describing objects. For any subset of attributes $B \subseteq C$, we can define an equivalence relation $IND(B)$ as follows.

$$IND(B) = \{(x, y) \in U \times U\colon a(x) = a(y),\ \forall a \in B\}.$$

Obviously, $IND(B) = \bigcap_{a \in B} IND(\{a\})$. By $[x]_B$ we denote the equivalence class of x with respect to $IND(B)$. For any subset $X \subseteq U$, $\underline{B}X = \{x \in U\colon [x]_B \subseteq X\}$ and $\overline{B}X = \{x \in U\colon [x]_B \cap X \neq \emptyset\}$ are called B-lower and B-upper approximations of X in IS, respectively.

An attribute $a \in B \subseteq C$ is superfluous in B if $IND(B) = IND(B - \{a\})$, otherwise a is indispensable in B. The collection of all indispensable attributes in C is called the core of IS. We say that $B \subseteq C$ is independent in IS if every attribute in B is indispensable in B. $B \subseteq C$ is called a reduct in IS if B is independent and $IND(B) = IND(C)$.

Fuzzy sets are another important extension of classical crisp sets [48,49]. Let U be a universal set. A fuzzy subset A of U is defined as a function assigning to each element x of U a value $A(x) \in [0, 1]$ and $A(x)$ is referred to as the membership degree of x to the fuzzy set A; we denote by $F(U)$ the set of all fuzzy subsets of U. For any $A, B \in F(U)$, we say that A is contained in B, denoted by $A \subseteq B$, if $A(x) \leqslant B(x)$ for all $x \in U$, and we say that $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$. The support of a fuzzy set A is a set defined as $\mathrm{supp}(A) = \{x \in U \mid A(x) > 0\}$. Given $A, B \in F(U)$, the union of A and B, denoted as $A \cup B$, is defined by $(A \cup B)(x) = A(x) \vee B(x)$ for all $x \in U$, where "$\vee$" is the maximum operation. The intersection of A and B, denoted as $A \cap B$, is given by $(A \cap B)(x) = A(x) \wedge B(x)$ for all $x \in U$, where "$\wedge$" is the minimum operation.

Let $U = \{x_1, x_2, \dots, x_n\}$ be a nonempty finite set of objects and $A_1, A_2, \dots, A_m$ be a family of fuzzy sets on U. If $\sum_{j=1}^{m} A_j(x_i) = 1$, $i = 1, 2, \dots, n$, then the family of fuzzy sets $\{A_j\colon\ j = 1, 2, \dots, m\}$ is called a fuzzy partition on U [50]. If $\sum_{j=1}^{m} A_j(x_i) \geqslant 1$, $i = 1, 2, \dots, n$, then $\{A_j\colon\ j = 1, 2, \dots, m\}$ is called a fuzzy covering on U [51].

3. Invariant properties of fuzzy mappings

In [48], Zadeh proposed the extension principle, which has become an important tool in fuzzy set theory and its applications. In this section, we introduce the notions of consistent and compatible mappings by Zadeh's extension principle and we construct a pair of rough fuzzy approximation operators by means of fuzzy mappings. We then examine some invariant properties of set operations of fuzzy sets.

Let U and V be two universes, $f : U \to V$ be a mapping from U to V. By Zadeh's extension principle, f can induce a fuzzy mapping from $F(U)$ to $F(V)$ and an inverse fuzzy mapping from $F(V)$ to $F(U)$, that is,

$$\hat{f} : F(U) \to F(V), \qquad A \mapsto \hat{f}(A) \in F(V), \quad \forall A \in F(U);$$

$$\hat{f}(A)(y) = \begin{cases} \sup_{x \in f^{-1}(y)} A(x), & y \in f(U); \\ 0, & y \notin f(U). \end{cases}$$

$$\hat{f}^{-1} : F(V) \to F(U), \qquad P \mapsto \hat{f}^{-1}(P) \in F(U), \quad \forall P \in F(V);$$

$$\hat{f}^{-1}(P)(x) = P\big(f(x)\big), \quad x \in U,$$

where $\hat{f}(A)$ is referred to as the image of the fuzzy set A and $f^{-1}(P)$ is referred to as the inverse image of the fuzzy set P. Under no confusion in subsequent discussion, we simply denote $\hat{f}$ and $\hat{f}^{-1}$ by f and f^{-1}, respectively.

The extension principle of fuzzy sets extends the concepts of mappings from classical sets to fuzzy power sets. Let $\mathbf{C} = \{A_1, A_2, \dots, A_m\}$ be a family of fuzzy sets on U, it is easily seen that $f(\mathbf{C}) = \{f(A_1), f(A_2), \dots, f(A_m)\}$ and $f(\mathbf{C})$ is also a family of fuzzy sets on U. Similarly, if $\mathbf{P} = \{P_1, P_2, \dots, P_m\}$ is a family of fuzzy sets on V, then $f^{-1}(\mathbf{P}) = \{f^{-1}(P_1), f^{-1}(P_2), \dots, f^{-1}(P_m)\}$ and $f^{-1}(\mathbf{P})$ is a family of fuzzy sets on V. In the following discussion, we always suppose that a given mapping $f : U \to V$ is surjective.

Proposition 3.1. *Let U and V be two universes, $f : U \to V$ be a mapping from U to V, $\mathbf{C}$ and $\mathbf{P}$ be families of fuzzy sets on U and V, respectively.*

(1) *If $\mathbf{C}$ is a fuzzy covering on U, then $f(\mathbf{C})$ is also a fuzzy covering on V.*
(2) *$\mathbf{P}$ is a fuzzy covering on V if and only if $f^{-1}(\mathbf{P})$ is a fuzzy covering on U. In particular, $\mathbf{P}$ is a fuzzy partition on V if and only if $f^{-1}(\mathbf{P})$ is a fuzzy partition on U.*

Proof. (1) Let $\mathbf{C} = \{A_1, A_2, \dots, A_m\}$. It follows from Zadeh's extension principle that $f(A_i)(y) = \sup_{x \in f^{-1}(y)} A_i(x)$ for any fuzzy set A_i and $y \in V$. This means that $\sum_{i=1}^{m} f(A_i)(y) = \sum_{i=1}^{m} \sup_{x \in f^{-1}(y)} A_i(x)$. Since $\mathbf{C} = \{A_1, A_2, \dots, A_m\}$ is a fuzzy covering on U, we have that $\sum_{i=1}^{m} A_i(x) \geqslant 1$ for any $x \in U$. Hence, $\sum_{i=1}^{m} f(A_i)(y) = \sum_{i=1}^{m} \sup_{x \in f^{-1}(y)} A_i(x) \geqslant \sum_{i=1}^{m} A_i(x) \geqslant 1$ for any $x \in f^{-1}(y)$. It follows that $\hat{f}(\mathbf{C})$ is also a fuzzy covering on V.

(2) Let $\mathbf{P} = \{P_1, P_2, \dots, P_m\}$. It follows from Zadeh's extension principle that $f^{-1}(P_i)(x) = P_i(f(x))$ for any fuzzy set P_i and $x \in U$. This means that $\sum_{i=1}^{m} f^{-1}(P_i)(x) = \sum_{i=1}^{m} P_i(f(x))$. This concludes the result. □

Remark 1. In general, if $f(\mathbf{C})$ is a fuzzy covering on V, we cannot ensure that $\mathbf{C}$ is a fuzzy covering on U. This means that it needs more conditions to ensure the inverse statement is true. Below we give an illustrative example.

Example 3.1. Let $U = \{x_1, x_2, x_3, x_4, x_5, x_6\}$, $V = \{y_1, y_2, y_3\}$, and

$$f: U \to V, \quad f(x) = \begin{cases} y_1, & x = x_1, x_2, x_3; \\ y_2, & x = x_4, x_5; \\ y_3, & x = x_6. \end{cases}$$

Let $\mathbf{C} = \{A_1, A_2, A_3\}$, $\mathbf{D} = \{B_1, B_2\}$, where $A_1 = \frac{0.4}{x_1} + \frac{0.6}{x_2} + \frac{0.5}{x_3} + \frac{0.1}{x_4} + \frac{0}{x_5} + \frac{0.3}{x_6}$, $A_2 = \frac{0.2}{x_1} + \frac{0.1}{x_2} + \frac{0.3}{x_3} + \frac{0.4}{x_4} + \frac{0.6}{x_5} + \frac{0.4}{x_6}$, $A_3 = \frac{0.6}{x_1} + \frac{0.3}{x_2} + \frac{0.2}{x_3} + \frac{0.5}{x_4} + \frac{0.4}{x_5} + \frac{0.4}{x_6}$, $B_1 = \frac{0.8}{x_1} + \frac{0.1}{x_2} + \frac{0.5}{x_3} + \frac{0.4}{x_4} + \frac{0.2}{x_5} + \frac{0.4}{x_6}$, $B_2 = \frac{0.1}{x_1} + \frac{0.5}{x_2} + \frac{0.4}{x_3} + \frac{0.4}{x_4} + \frac{0.7}{x_5} + \frac{0.7}{x_6}$. Obviously, $\mathbf{C}$ is a covering on U and $\mathbf{D}$ is not a covering on U.

By Zadeh's extension principle, we get

Hence, $f(A_1) = \frac{0.6}{y_1} + \frac{0.1}{y_2} + \frac{0.3}{y_3}$, $f(A_2) = \frac{0.3}{y_1} + \frac{0.6}{y_2} + \frac{0.4}{y_3}$, $f(A_3) = \frac{0.6}{y_1} + \frac{0.5}{y_2} + \frac{0.4}{y_3}$, $f(B_1) = \frac{0.8}{y_1} + \frac{0.4}{y_2} + \frac{0.4}{y_3}$, $f(B_2) = \frac{0.5}{y_1} + \frac{0.7}{y_2} + \frac{0.7}{y_3}$. It can be easily seen that $f(\mathbf{C})$ and $f(\mathbf{D})$ are fuzzy coverings on U. □

Definition 3.1. Let U and V be two universes, $f: U \to V$ a mapping from U to V, and $A_1, A_2 \in F(U)$. Let $[x]_f = \{y \in U: f(y) = f(x)\}$, then $\{[x]_f: x \in U\}$ is a partition on U. For any $x \in U$, if one of the following statements holds:

(1) $A_1(u) \leqslant A_2(u)$ for any $u \in [x]_f$,
(2) $A_1(u) \geqslant A_2(u)$ for any $u \in [x]_f$,

then f is called consistent with respect to A_1 and A_2. For any $x \in U$, if $A(u) = A(v)$ for any $u, v \in [x]_f$, then f is called compatible with respect to A.

From Definition 3.1, an injection is trivially a compatible mapping with respect to any fuzzy set A.

Proposition 3.2. *Let $f: U \to V$, $A, A_1, A_2 \in F(U)$. If f is compatible with respect to A, A_1 and A_2, respectively, then*

(1) *f is compatible with respect to $A_1 \cap A_2$;*
(2) *f is compatible with respect to $A_1 \cup A_2$;*
(3) *f is compatible with respect to the complement of A.*

Proof. Straightforward. □

It is obvious to see that a fuzzy mapping must be consistent with respect to two fuzzy sets if it is compatible with respect to the fuzzy sets, respectively. The following propositions and corollaries show that a consistent or compatible fuzzy mapping can preserve set operations of fuzzy sets.

Proposition 3.3. *Let $f: U \to V$ and $A_1, A_2 \in F(U)$. If f is consistent with respect to A_1 and A_2, then $f(A_1 \cap A_2) = f(A_1) \cap f(A_2)$.*

Proof. It is obvious that $f(A_1 \cap A_2) \subseteq f(A_1) \cap f(A_2)$ holds. In the following, we prove the inverse inclusion.

For any $y \in f(U)$, since f is consistent with respect to A_1 and A_2, it follows from Definition 3.1 that one of the following conditions holds:

$$(1) \quad A_1(x) \leqslant A_2(x), \qquad (2) \quad A_1(x) \geqslant A_2(x)$$

for any $x \in f^{-1}(y)$.

For case (1), we have

$$\begin{aligned} f(A_1 \cap A_2)(y) &= \sup_{x \in f^{-1}(y)} (A_1 \cap A_2)(x) \\ &= \sup_{x \in f^{-1}(y)} \{A_1(x) \wedge A_2(x)\} \\ &= \sup_{x \in f^{-1}(y)} A_1(x) \end{aligned}$$

and

$$\begin{aligned}\big(f(A_1)\cap f(A_2)\big)(y) &= f(A_1)(y)\wedge f(A_2)(y)\\ &= \Big(\sup_{x\in f^{-1}(y)} A_1(x)\Big)\wedge\Big(\sup_{x\in f^{-1}(y)} A_2(x)\Big)\\ &= \sup_{x\in f^{-1}(y)} A_1(x).\end{aligned}$$

Hence $f(A_1\cap A_2)=f(A_1)\cap f(A_2)$. Similarly, for case (2), we also have $f(A_1\cap A_2)=f(A_1)\cap f(A_2)$, therefore, we conclude the proof. □

By Proposition 3.3, we can obtain the following corollaries.

Corollary 3.4. *Let $f:U\to V, A_1, A_2,\dots,A_m\in F(U)$. If f is consistent with respect to any two of fuzzy sets $A_1, A_2,\dots,A_n$, then $f(\bigcap_{i=1}^m A_i)=\bigcap_{i=1}^m f(A_i)$ holds.*

Corollary 3.5. *Let $f:U\to V, A_1, A_2,\dots,A_m\in F(U)$. If f is compatible with respect to each of fuzzy sets $A_1, A_2,\dots,A_n$, then $f(\bigcap_{i=1}^m A_i)=\bigcap_{i=1}^m f(A_i)$ holds.*

Proposition 3.6. *Let $f:U\to V$ and $A\in F(U)$. If f is compatible with respect to A, then $f^{-1}(f(A))=A$.*

Proof. If f is compatible with respect to A, it follows from Definition 3.1 that $A(u)=A(x)$ for any $u\in f^{-1}(f(x))$. This means that

$$f^{-1}\big(f(A)\big)(x)=f(A)\big(f(x)\big)=\sup_{f(u)=f(x)} A(u)=A(x).$$

This concludes the proof. □

From Proposition 3.6, we can easily get the following corollary.

Corollary 3.7. *Let $f:U\to V$ and $\mathbf{C}=\{A_1, A_2,\dots,A_m\}$ be a family of fuzzy sets on U. If f is compatible with respect to A_i $(i\leqslant m)$, then $f^{-1}(f(\mathbf{C}))=\mathbf{C}$.*

Proposition 3.8. *Let $f:U\to V$ and $\mathbf{C}=\{A_1, A_2,\dots,A_m\}$ be a family of fuzzy sets on U. If f is compatible with respect to A_i $(i\leqslant m)$, then $\mathbf{C}$ is a fuzzy covering if and only if $f(\mathbf{C})$ is a fuzzy covering on V. In particular, $\mathbf{C}$ is a fuzzy partition if and only if $f(\mathbf{C})$ is a fuzzy partition on V.*

Proof. From Proposition 3.1, it is easily concluded that $f(\mathbf{C})$ is a fuzzy covering on V if $\mathbf{C}$ is a fuzzy covering. In the following, we prove the converse of the statement.

Let $f(\mathbf{C})=\{f(A)_1, f(A_2),\dots,f(A_m)\}$. Since $f(\mathbf{C})$ is a fuzzy covering on V, it follows from the concept of fuzzy covering that $\sum_{i=1}^m f(A_i)(y)\geqslant 1$ for any $y\in f(U)$. Let $y=f(x)$, from Proposition 3.6 we get that $f^{-1}(f(A))(x)=f(A)(f(x))=f(A)(y)=A(x)$.

Hence, $\sum_{i=1}^m A_i(x)\geqslant 1$, which implies that $\mathbf{C}$ is a fuzzy covering on U. □

In the following, we introduce the concepts of approximations of fuzzy sets based on the extension principle and examine some invariant properties of approximation operators under consistent and compatible mappings.

Definition 3.2. Let U and V be two universes, $f:U\to V$ be a mapping from U to V. For any fuzzy set $A\in F(U)$, we define a pair of lower and upper approximations of A with respect to f as follows:

$$\underline{apr}_f(A)(x)=\inf_{f(y)=f(x)} A(y),\quad x\in U,$$
$$\overline{apr}_f(A)(x)=\sup_{f(y)=f(x)} A(y),\quad x\in U.$$

$\underline{apr}_f$ and $\overline{apr}_f$ are referred to as lower and upper approximation operators of fuzzy sets with respect to the mapping f, respectively. The pair $(\underline{apr}_f(A), \overline{apr}_f(A))$ is called the rough fuzzy set of A. In the following, we simply denote $\underline{apr}_f$ and $\overline{apr}_f$ by $\underline{apr}$ and $\overline{apr}$, respectively.

Theorem 3.9. *Let $f: U \to V$ and $A, B \in F(U)$. Then $\underline{apr}$ and $\overline{apr}$ have the following properties*:

(1) $\underline{apr}(A) \subseteq A \subseteq \overline{apr}(A)$.
(2) $\underline{apr}(A \cap B) = \underline{apr}(A) \cap \underline{apr}(B)$, $\overline{apr}(A \cup B) = \overline{apr}(A) \cup \overline{apr}(B)$.
(3) $\underline{apr}(A) = \sim \overline{apr}(\sim A)$, $\overline{apr}(A) = \sim \underline{apr}(\sim A)$.
(4) $\underline{apr}(A \cup B) \supseteq \underline{apr}(A) \cup \underline{apr}(B)$, $\overline{apr}(A \cap B) \subseteq \overline{apr}(A) \cap \overline{apr}(B)$.
If f is consistent with respect to A and B, then the equalities hold.
(5) $\overline{apr}(\overline{apr}(A)) = \underline{apr}(\overline{apr}(A)) = \overline{apr}(A)$.
(6) $\overline{apr}(\underline{apr}(A)) = \underline{apr}(\underline{apr}(A)) = \underline{apr}(A)$.

Proof. (1), (2), (3), (5), (6) can be proved directly. Next, we give the proof of (4).

We only prove the first statement of (4), that is, $\underline{apr}(A \cup B) \supseteq \underline{apr}(A) \cup \underline{apr}(B)$ and the equality holds if f is consistent with respect to A and B. For the second statement, it is similar to the proof of the first statement.

For any $x \in U$, we have that $(A \cup B)(x) \geqslant A(x)$ and $(A \cup B)(x) \geqslant B(x)$. From Definition 3.2, we get that

$$\underline{apr}(A \cup B)(x) = \inf_{f(y)=f(x)} (A \cup B)(y) \geqslant \inf_{f(y)=f(x)} A(y) = \underline{apr}A(x)$$

and

$$\underline{apr}(A \cup B)(x) = \inf_{f(y)=f(x)} (A \cup B)(y) \geqslant \inf_{f(y)=f(x)} B(y) = \underline{apr}B(x).$$

This means that $\underline{apr}(A \cup B) \supseteq \underline{apr}(A) \cup \underline{apr}(B)$.

If f is consistent with respect to A and B, it follows from Definition 3.1 that one of the following conditions holds:

$$(1) \quad A(y) \leqslant B(y), \qquad (2) \quad A(y) \geqslant B(y)$$

for any $y \in f^{-1}(f(x))$.

For case (1), we have

$$\underline{apr}(A \cup B)(x) = \inf_{f(y)=f(x)} (A \cup B)(y) = \inf_{f(y)=f(x)} B(y) = \underline{apr}B(x)$$

and

$$\underline{apr}B(x) = \inf_{f(y)=f(x)} B(y) \geqslant \inf_{f(y)=f(x)} A(y) = \underline{apr}A(x).$$

Thus, $\underline{apr}(A \cup B) = \underline{apr}(A) \cup \underline{apr}(B)$. Similarly, for case (2), we also have $\underline{apr}(A \cup B) = \underline{apr}(A) \cup \underline{apr}(B)$. Therefore, we conclude the proof. □

Corollary 3.10. *Let $f: U \to V$ and $A, B \in F(U)$. If f is compatible with respect to both A and B, then $\underline{apr}(A \cup B) = \underline{apr}(A) \cup \underline{apr}(B)$, $\overline{apr}(A \cap B) = \overline{apr}(A) \cap \overline{apr}(B)$.*

4. Homomorphism between fuzzy information systems

By means of the results of the above sections, we introduce the notions of homomorphisms between fuzzy information systems and investigate some properties of fuzzy information systems under the condition of homomorphisms. Let us start with introducing the notions of fuzzy information systems.

Definition 4.1. Let U and V be finite universes, $f: U \to V$ a surjective mapping from U to V, and $\mathbf{C} = \{A_1, A_2, \ldots, A_m\}$ a family of fuzzy sets on U. Then the pair $(U, \mathbf{C})$ is referred to as a fuzzy information system, and the pair $(V, f(\mathbf{C}))$ is referred to as an f-induced fuzzy information system of $(U, \mathbf{C})$.

Definition 4.2. Let $(U, \mathbf{C})$ be a fuzzy information system and $(V, f(\mathbf{C}))$ an f-induced fuzzy information system. f is referred to as a homomorphism from $(U, \mathbf{C})$ to $(V, f(\mathbf{C}))$ if f satisfies the following conditions:

$$(1)\quad f(\cap\mathbf{C}) = \cap f(\mathbf{C}); \qquad (2)\quad f(\cup\mathbf{C}) = \cup f(\mathbf{C}).$$

Proposition 4.1. *Let $(U, \mathbf{C})$ be a fuzzy information system and $(V, f(\mathbf{C}))$ an f-induced fuzzy information system. If f is compatible with respect to any fuzzy set $A \in \mathbf{C}$, then f is a homomorphism from $(U, \mathbf{C})$ to $(V, f(\mathbf{C}))$.*

Proof. The condition about the intersection in Definition 4.2 follows immediately from Corollary 3.5. The condition about the union follows from Zadeh's extension principle. □

Proposition 4.2. *Let $(U, \mathbf{C})$ be a fuzzy information system and $(V, f(\mathbf{C}))$ an f-induced fuzzy information system. If for any $A_i, A_j \in \mathbf{C}$, f is consistent with respect to A_i and A_j, then f is a homomorphism from $(U, \mathbf{C})$ to $(V, f(\mathbf{C}))$.*

Proof. The condition about the intersection in Definition 4.2 follows immediately from Corollary 3.5. The condition about the union follows from Zadeh's extension principle. □

In order to identify homomorphism satisfying different conditions, we make the following appointments.

Definition 4.3. Let $(U, \mathbf{C} = \{A_1, A_2, \ldots, A_n\})$ be a fuzzy information system and $(V, f(\mathbf{C}))$ an f-induced fuzzy information system. If for any $A_i, A_j \in \mathbf{C}$, f is consistent with respect to A_i and A_j, then *f* is called a consistency-based homomorphism. If *f* is compatible with respect to any fuzzy set A_i $(i \leqslant n)$, then f is called a compatibility-based homomorphism.

From Definition 3.1 and Definition 4.2, Corollaries 3.4 and 3.5, and Proposition 4.2, we immediately get the following corollary.

Corollary 4.3. *Let $(U, \mathbf{C})$ be a fuzzy information system and $(V, f(\mathbf{C}))$ an f-induced fuzzy information system. If f is a compatibility-based homomorphism, then f is a consistency-based homomorphism.*

Remark 2. After introducing the notion of a homomorphism, all the propositions and corollaries in Section 3 may be viewed as the statements about properties of homomorphisms between fuzzy information systems.

Definition 4.4. Let $(U, \mathbf{C})$ be a fuzzy information system, $A \in \mathbf{C}$ and $\mathbf{P} \subseteq \mathbf{C}$. A is said to be L-superfluous in $\mathbf{C}$ if $\cap\mathbf{C} = \cap(\mathbf{C} - \{A\})$, otherwise it is L-indispensable. The subset $\mathbf{P}$ is referred to as a lower reduct of $\mathbf{C}$ if $\mathbf{P}$ satisfies the following conditions: (1) $\cap\mathbf{P} = \cap\mathbf{C}$; (2) $\forall A_i \in \mathbf{P}, \cap\mathbf{P} \subset \cap(\mathbf{P} - \{A_i\})$.

Definition 4.5. Let $(U, \mathbf{C})$ be a fuzzy information system, $A \in \mathbf{C}$ and $\mathbf{P} \subseteq \mathbf{C}$. A is said to be H-superfluous in $\mathbf{C}$ if $\cup\mathbf{C} = \cup(\mathbf{C} - \{A\})$, otherwise it is H-indispensable. The subset $\mathbf{P}$ is referred to as an upper reduct of $\mathbf{C}$ if $\mathbf{P}$ satisfies the following conditions: (1) $\cup\mathbf{P} = \cup\mathbf{C}$; (2) $\forall A_i \in \mathbf{P}, \cup\mathbf{P} \subset \cup(\mathbf{P} - \{A_i\})$.

Definition 4.6. Let $(U, \mathbf{C})$ be a fuzzy information system and $\mathbf{P} \subseteq \mathbf{C}$. If $\mathbf{P}$ is both an upper and a lower reduct of $\mathbf{C}$, then $\mathbf{P}$ is referred to as a reduct of $\mathbf{C}$.

The following theorem and corollary show that the attribute reductions of an original information system and its image system are equivalent to each other under the condition of a consistency-based homomorphism or a compatibility-based homomorphism.

Theorem 4.4. *Let $(U, \mathbf{C})$ be a fuzzy information system, $(V, f(\mathbf{C}))$ an f-induced fuzzy information system, f a consistency-based homomorphism from $(U, \mathbf{C})$ to $(V, f(\mathbf{C}))$ and $\mathbf{P} \subseteq \mathbf{C}$. Then $\mathbf{P}$ is a reduct of $\mathbf{C}$ if and only if $f(\mathbf{P})$ is a reduct of $f(\mathbf{C})$.*

Proof. We only need to prove that $\mathbf{P}$ is a lower (upper) reduct of $\mathbf{C}$ if and only if $f(\mathbf{P})$ is a lower (upper) reduct of $f(\mathbf{C})$. We first prove that $\mathbf{P}$ is a lower reduct of $\mathbf{C}$ if and only if $f(\mathbf{P})$ is a lower reduct of $f(\mathbf{C})$.

Assume that $\mathbf{P}$ is a lower reduct of $\mathbf{C}$. Hence, we have $\cap\mathbf{P} = \cap\mathbf{C}$. Thus $f(\cap\mathbf{P}) = f(\cap\mathbf{C})$. Since f is a consistency-based homomorphism from $(U, \mathbf{C})$ to $(V, f(\mathbf{C}))$, by Definition 4.2 and Corollary 3.4, we have $\cap f(\mathbf{P}) = \cap f(\mathbf{C})$. Assume that there exists $A_i \in \mathbf{P}$ such that $\cap(f(\mathbf{P}) - f(A_i)) = \cap f(\mathbf{P})$. Then we have that $\cap f(\mathbf{P} - \{A_i\}) = \cap f(\mathbf{P}) = \cap f(\mathbf{C})$. Similarly, by Definition 4.2 and Corollary 3.4, it follows that $f(\cap(\mathbf{P} - \{A_i\})) = f(\cap\mathbf{C})$. Again, since $\mathbf{P}$ is a lower reduct of $\mathbf{C}$, we have $\cap\mathbf{P} \neq \cap(\mathbf{P} - \{A_i\})$. Thus there must be $x_0 \in U$ such that $\cap\mathbf{P}(x_0) < \cap(\mathbf{P} - \{A_i\})(x_0)$, which implies

$$\begin{aligned} f\big(\cap\big(\mathbf{P} - \{A_i\}\big)\big)\big(f(x_0)\big) &= \sup_{u \in f^{-1}(f(x_0))} \cap\big(\mathbf{P} - \{A_i\}\big)(u) \\ &> \sup_{u \in f^{-1}(f(x_0))} \cap\mathbf{P}(u) \\ &= f(\cap\mathbf{P})\big(f(x_0)\big) \\ &= f(\cap\mathbf{C})\big(f(x_0)\big). \end{aligned}$$

This is a contradiction.

Let $f(\mathbf{P})$ be a lower reduct of $f(\mathbf{C})$. Then $\cap f(\mathbf{P}) = \cap f(\mathbf{C})$. Since f is a consistency-based homomorphism from $(U, \mathbf{C})$ to $(V, f(\mathbf{C}))$, by Definition 4.2 and Corollary 3.4, we have $f(\cap\mathbf{P}) = f(\cap\mathbf{C})$. Assume that $\cap\mathbf{P} \supsetneq \cap\mathbf{C}$, there must exist $x_0 \in U$ such that $\cap\mathbf{C}(x_0) < \cap\mathbf{P}(x_0)$. Then we have

$$\begin{aligned} f(\cap\mathbf{P})\big(f(x_0)\big) &= \sup_{u \in f^{-1}(f(x_0))} \cap\mathbf{P}(u) \\ &> \sup_{u \in f^{-1}(f(x_0))} \cap\mathbf{C}(u) \\ &= f(\cap\mathbf{C})\big(f(x_0)\big). \end{aligned}$$

This is a contradiction. Thus $\cap\mathbf{P} = \cap\mathbf{C}$. Assume that there exists $A_i \in \mathbf{P}$ such that $\cap(\mathrm{P} - \{A_i\}) = \cap\mathbf{C}$. Then $f(\cap(\mathbf{P} - \{A_i\})) = f(\cap\mathbf{C})$. Again, by Definition 4.2 and Corollary 3.4, we have $\cap f(\mathbf{P} - \{A_i\}) = \cap f(\mathbf{C})$. Hence, $\cap(f(\mathbf{P}) - f(A_i)) = \cap f(\mathbf{C})$. This is a contradiction to that $f(\mathbf{P})$ is a lower reduct of $f(\mathbf{C})$.

Similar to the proof of the statement that $\mathbf{P}$ is a lower reduct of $\mathbf{C}$ if and only if $f(\mathbf{P})$ is a lower reduct of $f(\mathbf{C})$, we can easily prove that $\mathbf{P}$ is an upper reduct of $\mathbf{C}$ if and only if $f(\mathbf{P})$ is an upper reduct of $f(\mathbf{C})$. Hence, we complete the proof of this theorem. □

By Theorem 4.4 and Definition 4.2, we immediately get the following corollary.

Corollary 4.5. *Let $(U, \mathbf{C})$ be a fuzzy information system and f a compatibility-based homomorphism from $(U, \mathbf{C})$ to $(V, f(\mathbf{C}))$, $\mathbf{P} \subseteq \mathbf{C}$. Then $\mathbf{P}$ is a reduct of $\mathbf{C}$ if and only if $f(\mathbf{P})$ is a reduct of $f(\mathbf{C})$.*

The following example is employed to illustrate our idea given in Theorem 4.4.

Example 4.1. Now we consider a decision-making problem with multiple attributes. Let $U = \{x_1, x_2, \dots, x_9\}$ be a set of samples, and $\mathbf{C} = \{A_1, A_2, A_3, A_4\}$ be a set of decision attributes, then $(U, \mathbf{C})$ is a decision information system. If samples take values in the interval [0, 1] on each decision attribute, then each decision attribute can define a fuzzy set on the sample set. Suppose that the four fuzzy sets are listed as follows.

$$\begin{aligned} A_1 &= \frac{1}{x_1} + \frac{0.2}{x_2} + \frac{0.4}{x_3} + \frac{0.8}{x_4} + \frac{0.1}{x_5} + \frac{0.3}{x_6} + \frac{0.7}{x_7} + \frac{0.3}{x_8} + \frac{0.5}{x_9}, \\ A_2 &= \frac{0.3}{x_1} + \frac{0.8}{x_2} + \frac{0.5}{x_3} + \frac{0.5}{x_4} + \frac{0.6}{x_5} + \frac{0.4}{x_6} + \frac{0.6}{x_7} + \frac{0.5}{x_8} + \frac{0.7}{x_9}, \\ A_3 &= \frac{0.5}{x_1} + \frac{0.8}{x_2} + \frac{0.9}{x_3} + \frac{0.8}{x_4} + \frac{0.5}{x_5} + \frac{1}{x_6} + \frac{0.6}{x_7} + \frac{0.3}{x_8} + \frac{0.8}{x_9}, \\ A_4 &= \frac{0.2}{x_1} + \frac{0.9}{x_2} + \frac{0.7}{x_3} + \frac{0.3}{x_4} + \frac{0.7}{x_5} + \frac{0.8}{x_6} + \frac{0.4}{x_7} + \frac{1}{x_8} + \frac{0.7}{x_9}. \end{aligned}$$

Assume that each of decision attributes is of the same importance. If we want to combine the multiple decisions into one comprehensive decision, we can employ two set operations on these fuzzy sets. One is intersection operation. The comprehensive decision induced by intersection operation is called positive decision. Another set operation is union of sets. The comprehensive decision induced by union operation is called possible decision. Then the positive decision and possible decision are computed as follows.

Positive decision:

$$A_1 \cap A_2 \cap A_3 \cap A_4 = \frac{0.2}{x_1} + \frac{0.2}{x_2} + \frac{0.4}{x_3} + \frac{0.3}{x_4} + \frac{0.1}{x_5} + \frac{0.3}{x_6} + \frac{0.4}{x_7} + \frac{0.3}{x_8} + \frac{0.5}{x_9},$$

Possible decision:

$$A_1 \cup A_2 \cup A_3 \cup A_4 = \frac{1}{x_1} + \frac{0.9}{x_2} + \frac{0.9}{x_3} + \frac{0.8}{x_4} + \frac{0.7}{x_5} + \frac{1}{x_6} + \frac{0.7}{x_7} + \frac{1}{x_8} + \frac{0.8}{x_9}.$$

Let $V = \{y_1, y_2, y_3\}$. Define a mapping as follows:

$$f : U \to V, \quad f(x) = \begin{cases} y_1, & x = x_1, x_4, x_7; \\ y_2, & x = x_2, x_5, x_8; \\ y_3, & x = x_3, x_6, x_9. \end{cases}$$

It is easy to verify that $f : U \to V$ is a consistent mapping with respect to any pair of fuzzy sets from A_1, A_2, A_3, A_4.

Let $f(\mathbf{C}) = \{f(A_1), f(A_2), f(A_3), f(A_4)\}$, where $f(A_1) = \frac{1}{y_1} + \frac{0.3}{y_2} + \frac{0.5}{y_3}$, $f(A_2) = \frac{0.6}{y_1} + \frac{0.8}{y_2} + \frac{0.7}{y_3}$, $f(A_3) = \frac{0.8}{y_1} + \frac{0.8}{y_2} + \frac{1}{y_3}$, $f(A_4) = \frac{0.4}{y_1} + \frac{1}{y_2} + \frac{0.8}{y_3}$. Then $(V, f(\mathbf{C}))$ is the image system of $(U, \mathbf{C})$. The positive and possible decisions of the image system are computed as follows:

$$f(A_1) \cap f(A_2) \cap f(A_3) \cap f(A_4) = \frac{0.4}{y_1} + \frac{0.3}{y_2} + \frac{0.5}{y_3},$$

$$f(A_1) \cup f(A_2) \cup f(A_3) \cup f(A_4) = \frac{1}{y_1} + \frac{1}{y_2} + \frac{1}{y_3}.$$

From Definition 4.3, we know that f is a consistency-based homomorphism from $(U, \mathbf{C})$ to $(V, f(\mathbf{C}))$. It is obvious to see that the original decision information system $(U, \mathbf{C})$ is relatively complex and its image system $(V, f(\mathbf{C}))$ is simpler. However, the original system and its image system have the following equivalent results.

(1) $\{f(A_2), f(A_3)\}$ is L-superfluous in $f(\mathbf{C}) \Leftrightarrow \{A_2, A_3\}$ is L-superfluous in $\mathbf{C}$, and $\{f(A_1)f(A_4)\}$ is a lower reduct of $f(\mathbf{C}) \Leftrightarrow \{A_1, A_4\}$ is a lower reduct of $\mathbf{C}$.

(2) $f(A_2)$ is H-superfluous in $f(\mathbf{C}) \Leftrightarrow A_2$ is H-superfluous in $\mathbf{C}$, and $\{f(A_1), f(A_3), f(A_4)\}$ is an upper reduct of $f(\mathbf{C}) \Leftrightarrow \{A_1, A_2, A_4\}$ is an upper reduct of $\mathbf{C}$.

(3) $\{f(A_1), f(A_3), f(A_4)\}$ is a reduct of $f(\mathbf{C}) \Leftrightarrow \{A_1, A_2, A_4\}$ is a reduct of $\mathbf{C}$.

These results mean that the reductions of the original system and image system are equivalent to each other. The image system $(V, f(\mathbf{C}))$ is simpler, but it has the equivalent reducts to the original decision system $(U, \mathbf{C})$. Therefore, we can reduce the complex original system by reducing its simpler image system.

If these nine samples are the training samples, then we have positive decision references $\{A_1, A_4\}$ and possible decision references $\{A_1, A_2, A_4\}$ for other input samples. It is clear that the decision attributes A_1 and A_4 are key attributes for combining multiple decisions into one comprehensive decision. □

Example 4.1 shows it is possible to reduce the complex original system by reducing the simpler homomorphic image system and so combine multiple decisions into one comprehensive decision. However, the feasibility and validness of this idea should be further examined by more practical experiments and will be our future work.

5. Conclusion and future work

In this work, we point out that a fuzzy mapping can preserve some basic set operations under consistence and compatibility of fuzzy sets. For a fuzzy information system, we can consider it as a combination of some fuzzy sets on the same universe. A fuzzy mapping between two universes can be explained as a fuzzy mapping between fuzzy information systems. A homomorphism is a special fuzzy mapping between two fuzzy information systems. Under the condition of homomorphism, we discuss some invariant properties of fuzzy information systems, and find out that the attribute reductions of an original information system and its image system are equivalent to each other. These results may have potential applications in knowledge reduction, decision making and reasoning about data. Our results also illustrate that some properties of an information system can be guaranteed in an explanation system, i.e., an information system gains acknowledgment from another information system. However, some results in this paper are right only under some sufficient conditions. How to explore the sufficient and necessary conditions of these results is our future work.

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