Chapter 7
Wavelets and Multiresolution Processing
● Background

● Multiresolution Expansions

● Wavelet Transforms in One Dimension

● Wavelet Transforms in Two Dimensions
FIGURE 7.1 A natural image and its local histogram variations.
- Image Pyramids
- Subband Coding
- The Haar Transform
The total number of elements in a $P+1$ level pyramid for $P>0$ is

$$N^2 \left( 1 + \frac{1}{4^1} + \frac{1}{4^2} + \cdots + \frac{1}{4^P} \right) \leq \frac{4}{3} N^2$$
FIGURE 7.2 (a) A pyramidal image structure and (b) system block diagram for creating it.
FIGURE 7.3 Two image pyramids and their statistics: (a) a Gaussian (approximation) pyramid and (b) a Laplacian (prediction residual) pyramid.
- Image Pyramids
- Subband Coding
- The Haar Transform
FIGURE 7.4 (a) A two-band filter bank for one-dimensional subband coding and decoding, and (b) its spectrum splitting properties.
The Z-transform, a generalization of the discrete Fourier transform, is the ideal tool for studying discrete-time, sampled-data systems.

The Z-transform of sequence $x(n)$ for $n=0,1,2,…$ is

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n}$$

Where $z$ is a complex variable.
● Downsampling by a factor of 2 in the time domain corresponds to the simple Z-domain operation

\[ x_{\text{down}}(n) = x(2n) \iff X_{\text{down}}(z) = \frac{1}{2} \left[ X(z^{1/2}) + X(-z^{1/2}) \right] \tag{7.1-2} \]

● Upsampling—again by a factor of 2—is defined by the transform pair

\[
x_{\text{up}}(n) = \begin{cases} 
  x(n/2) & \text{if } n = 0, 2, 4, \ldots \\
  0 & \text{otherwise}
\end{cases} \iff X_{\text{up}}(z) = x(z^2) \tag{7.1-3}
\]
If sequence $x(n)$ is downsampled and subsequently upsampled to yield $y(n)$, Eqs.(7.1-2) and (7.1-3) combine to yield

$$\hat{X}(z) = \frac{1}{2} [X(z) + X(-z)]$$

where $\hat{x}(n) = Z^{-1}[\hat{X}(z)]$ is the downsampled-upsampled sequence.

Its inverse Z-transform is

$$Z^{-1}[X(-z)] = (-1)^n x(n)$$
We can express the system’s output as

\[
\hat{X}(z) = \frac{1}{2} G_0(z) [H_0(z)X(z) + H_0(-z)X(-z)] \\
+ \frac{1}{2} G_1(z) [H_1(z)X(z) + H_1(-z)X(-z)]
\]

The output of filter \( h_0(n) \) is defined by the transform pair

\[
h_0(n) * x(n) = \sum_k h_0(n - k)x(k) \leftrightarrow H_0(z)X(z)
\]
As with Fourier transform, convolution in the time (or spatial domain is equivalent to multiplication in the Z-domain.

\[ \hat{X}(z) = \frac{1}{2} \left[ H_0(z)G_0(z) + H_1(z)G_1(z) \right] X(z) \]

\[ + \frac{1}{2} \left[ H_0(-z)G_0(z) + H_1(-z)G_1(z) \right] X(-z) \]
For error-free reconstruction of the input, \( \hat{x}(n) = x(n) \) and \( \hat{X}(z) = X(z) \). Thus, we impose the following conditions:

\[
H_0(-z)G_0(z) + H_1(-z)G_1(z) = 0 \\
H_0(z)G_0(z) + H_1(z)G_1(z) = 2
\]

To get

\[
\begin{bmatrix}
G_0(z) \\
G_1(z)
\end{bmatrix} = \frac{2}{\det(H_m(z))} \begin{bmatrix}
H_1(-z) \\
- H_0(-z)
\end{bmatrix}
\]

Where \( \det(H_m(z)) \) denotes the determinant of \( H_m(z) \).
\[
\det(H_m(z)) = \alpha z^{-(2k+1)}
\]

- Letting \( \alpha = 2 \), and taking the inverse Z-transform, we get
  \[
g_0(n) = (-1)^n h_1(n)
\]
  \[
g_1(n) = (-1)^{n+1} h_0(n)
\]

- Letting \( \alpha = -2 \), and taking the inverse Z-transform, we get
  \[
g_0(n) = (-1)^{n+1} h_1(n)
\]
  \[
g_1(n) = (-1)^n h_0(n)
\]
Three general solutions:

- Quadrature mirror filters (OMFs)
- Conjugate quadrature filters (CQFs)
- Orthonormal

<table>
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<th>Filter</th>
<th>QMF</th>
<th>CQF</th>
<th>Orthonormal</th>
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<td>$H_0(z)$</td>
<td>$H_0^2(z) - H_0^2(-z) = 2$</td>
<td>$H_0(z)H_0(z^{-1}) + H_0^2(-z)H_0(-z^{-1}) = 2$</td>
<td>$G_0(z^{-1})$</td>
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<tr>
<td>$H_1(z)$</td>
<td>$H_0(-z)$</td>
<td>$z^{-1}H_0(-z^{-1})$</td>
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<td>$G_0(z)$</td>
<td>$H_0(z)$</td>
<td>$H_0(z^{-1})$</td>
<td>$G_0(z)G_0(z^{-1}) + G_0(-z)G_0(-z^{-1}) = 2$</td>
</tr>
<tr>
<td>$G_1(z)$</td>
<td>$-H_0(-z)$</td>
<td>$zH_0(-z)$</td>
<td>$-z^{-2K+1}G_0(-z^{-1})$</td>
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FIGURE 7.5 A two-dimensional, four-band filter bank for subband image coding.
FIGURE 7.6 The impulse responses of four 8-tap Daubechies orthonormal filters.
FIGURE 7.7 A four-band split of the vase in Fig. 7.1 using the subband coding system of Fig. 7.5.
- Image Pyramids
- Subband Coding
- The Haar Transform
The Haar transform can be expressed in matrix form

\[ T = HFH^T \]

Where

- F is an N*N image matrix,
- H is an N*N transformation matrix,
- T is the resulting N*N transform.
For the Haar transform, transformation matrix $H$ contains the Haar basis functions, $h_k(z)$. They are defined over the continuous, closed interval $z \in [0,1]$ for $k=0,1,2,…,N-1$, where $N = 2^n$.

To generate $H$, we define the integer $k$ such that

$$k = 2^p + q - 1$$

where $0 \leq p \leq n-1$, $q = 0$ or $1$ for $p = 0$.

$$1 \leq q \leq 2^p \quad \text{for} \quad p \neq 0$$
Then the Haar basis functions are

\[ h_0(z) = h_{00}(z) = \frac{1}{\sqrt{N}}, \quad z \in [0,1] \]

and

\[ h_k(z) = h_{pq}(z) = \frac{1}{\sqrt{N}} \begin{cases} 2^{p/2} & (q - 1)/2^p \leq z < (q - 0.5)/2^p \\ -2^{p/2} & (q - 0.5)/2^p \leq z < q/2^p \\ 0 & otherwise, \ z \in [0,1] \end{cases} \]
The ith row of an N*N Haar transformation matrix contains the elements of

$$h_i(z) \text{ for } z = 0/N, 1/N, 2/N, \ldots, (N-1)/N.$$ 

If N=4, for example k,q, and p assume the values

<table>
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<tr>
<th>k</th>
<th>p</th>
<th>q</th>
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<tr>
<td>0</td>
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<td>0</td>
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<tr>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>2</td>
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</table>
The 4*4 transformation matrix, $H_4$, is

$$H_4 = \frac{1}{\sqrt{4}} \begin{bmatrix}
1 & 1 & 1 & 1 \\
1 & 1 & -1 & -1 \\
\sqrt{2} & -\sqrt{2} & 0 & 0 \\
0 & 0 & \sqrt{2} & -\sqrt{2}
\end{bmatrix}$$
FIGURE 7.8 (a) A discrete wavelet transform using Haar basis functions. Its local histogram variations are also shown:
(b)–(d) Several different approximations (64 × 64, 128 × 128, and 256 × 256) that can be obtained from (a).
- Background
- Multiresolution Expansions
- Wavelet Transforms in One Dimension
- Wavelet Transforms in Two Dimensions
- Series Expansion
- Scaling Functions
- Wavelet Functions
A signal of function $f(x)$ can often be better analyzed as a linear combination of expansion functions

$$f(x) = \sum_{k} \alpha_k \varphi_k(x)$$

- $k$ is an integer index of the finite or infinite sum;
- $\alpha_k$ are real-valued expansion coefficients;
- $\varphi_k(x)$ are real-valued expansion functions.
These coefficients are computed by taking the integral inner products of the dual \( \tilde{\phi}_k(x) \)'s and function \( f(x) \). That is

\[
\alpha_k = \langle \tilde{\phi}_k(x), f(x) \rangle = \int \tilde{\phi}_k^*(x) f(x) dx
\]
- Series Expansion
- Scaling Functions
- Wavelet Functions
The set of expansion functions composed of integer translations and binary scaling of the real, square-integrable function $\varphi(x)$; that is, the set $\{\varphi_{j,k}(x)\}$ where

$$\varphi_{j,k}(x) = 2^{j/2} \varphi(2^j x - k)$$
If $f(x) \in V_{j_0}$, it can be written

$$f(x) = \sum_k \alpha_k \varphi_{j_0,k}(x)$$

We will denote the subspace spanned over $k$ for any $j$ as

$$V_j = \text{Span}\{\varphi_{j,k}(x)\}$$


\[ \varphi_{0,0}(x) = \varphi(x) \]

\[ \varphi_{0,1}(x) = \varphi(x - 1) \]

\[ \varphi_{1,0}(x) = \sqrt{2} \varphi(2x) \]

\[ \varphi_{1,1}(x) = \sqrt{2} \varphi(2x - 1) \]

\[ f(x) = V_1 \]

\[ \varphi_{0,0}(x) \in V_1 \]

\[ \varphi_{1,1} \]

\[ 0.5 \varphi_{1,0} \]

\[ -0.25 \varphi_{1,4} \]

\[ \varphi_{1,0}/\sqrt{2} \]

\[ \varphi_{1,1}/\sqrt{2} \]

**FIGURE 7.9** Haar scaling functions in \( V_0 \) in \( V_1 \).
The simple scaling function in the preceding example obeys the four fundamental requirements of multiresolution analysis:

- MRA Requirement 1: The scaling function is orthogonal to its integer translates;
- MRA Requirement 2: The subspaces spanned by the scaling function at low scales are nested within those spanned at higher scales.
FIGURE 7.10 The nested function spaces spanned by a scaling function.
- MRA Requirement 3: The only function that is common to all $V_j$ is $f(x)=0$.

- MRA Requirement 4: Any function can be represented with arbitrary precision.
- Series Expansion
- Scaling Functions
- Wavelet Functions
Given a scaling function that meets the MRA requirements of the previous section, we can define a wavelet function $\psi(x)$ that, together with its integer translates and binary scaling, spans the difference between any two adjacent scaling subspaces, $V_j$ and $V_{j+1}$. We define the set $\{\psi_{j,k}(x)\}$ of wavelets

$$\left\{\psi_{j,k}(x) = 2^{j/2} \psi(2^j x - k)\right\}$$
As with scaling functions, we write

\[ W_j = \text{Span}\{\psi_{j,k}(x)\} \]

And note that if \( f(x) \in W_j \)

\[ f(x) = \sum_k \alpha_k \psi_{j,k}(x) \]

The scaling and wavelet function subspaces are related by

\[ V_{j+1} = V_j \oplus W_j \]
We can now express the space of all measurable, square-integrable functions as

\[ L^2(R) = V_0 \oplus W_0 \oplus W_1 \oplus \ldots \]

or

\[ L^2(R) = V_1 \oplus W_1 \oplus W_2 \oplus \ldots \]
\[ V_2 = V_1 \oplus W_1 = V_0 \oplus W_0 \oplus W_1 \]

\[ V_1 = V_0 \oplus W_0 \]

**Figure 7.11** The relationship between scaling and wavelet function spaces.
The Haar wavelet function is

\[ \psi(x) = \begin{cases} 
1 & 0 \leq x < 0.5 \\
-1 & 0.5 \leq x < 1 \\
0 & \text{elsewhere}
\end{cases} \]
\( \psi(x) = \psi_{0,0}(x) \)

\( \psi_{0,2}(x) = \psi(x-2) \)

\( \psi_{1,0}(x) = \sqrt{2}\psi(2x) \)

\( f(x) \in V_1 = V_0 \oplus W_0 \)

\( f_a(x) \in V_0 \)

\( f_a(x) \in W_0 \)

**FIGURE 7.12** Haar wavelet functions in \( W_0 \) and \( W_1 \).
- Background
- Multiresolution Expansions
- Wavelet Transforms in One Dimension
- Wavelet Transforms in Two Dimensions
● The Wavelet Series Expansions
● The Discrete Wavelet Transform
● The Continuous Wavelet Transform
Defining the wavelet series expansion of function \( f(x) \in L^2(R) \) relative to wavelet \( \psi(x) \) and scaling function \( \phi(x) \). \( f(x) \) can be written as

\[
f(x) = \sum_{k} c_{j_0}(k) \phi_{j_0,k}(x) + \sum_{j=j_0}^{\infty} d_j(k) \psi_{j,k}(x)
\]

- \( c_{j_0}(k)'s \) : the approximation or scaling coefficients;
- \( d_j(k)'s \) : the detail or wavelet coefficients.
If the expansion functions form an orthonormal basis or tight frame, the expansion coefficients are calculated as

$$c_{j_0}(k) = \langle f(x), \varphi_{j_0,k}(x) \rangle = \int f(x) \varphi_{j_0,k}(x) dx$$

and

$$d_j(k) = \langle f(x), \psi_{j,k}(x) \rangle = \int f(x) \psi_{j,k}(x) dx$$
FIGURE 7.13 A wavelet series expansion of $y = x^2$ using Haar wavelets.
The Wavelet Series Expansions

The Discrete Wavelet Transform

The Continuous Wavelet Transform
If the function being expanded is a sequence of numbers, like samples of a continuous function \( f(x) \), the resulting coefficients are called the discrete wavelet transform (DWT) of \( f(x) \).

\[
W_{\phi}(j_0, k) = \frac{1}{\sqrt{M}} \sum_x f(x) \phi_{j_0, k}(x)
\]

\[
W_{\psi}(j, k) = \frac{1}{\sqrt{M}} \sum_x f(x) \psi_{j, k}(x)
\]

and

\[
f(x) = \frac{1}{\sqrt{M}} \sum_k W_{\phi}(j_0, k) \phi_{j_0, k}(x) + \frac{1}{\sqrt{M}} \sum_{j=j_0}^{\infty} \sum_k W_{\psi}(j, k) \psi_{j, k}(x)
\]
Consider the discrete function of four points:
\[ f(0)=1, \ f(1)=4, \ f(2)=-3, \ \text{and} \ f(3)=0 \]

Since M=4, J=2 and, with \( j_0=0 \), the summations are performed over
\[ x=0,1,2,3, \]
\[ j=0,1, \ \text{and} \]
\[ k=0 \ \text{for} \ j=0 \]
or \( k=0,1 \ \text{for} \ j=1 \).
We find that

\[ W_\varphi (0,0) = \frac{1}{2} \sum_{x=0}^{3} f(x) \varphi_{0,0}(x) = \frac{1}{2} \left[ 1 \cdot 1 + 4 \cdot 1 - 3 \cdot 1 + 0 \cdot 1 \right] = 1 \]

\[ W_\psi (0,0) = \frac{1}{2} \left[ 1 \cdot 1 + 4 \cdot 1 - 3 \cdot (-1) + 0 \cdot (-1) \right] = 4 \]

\[ W_\psi (1,0) = \frac{1}{2} \left[ 1 \cdot \sqrt{2} + 4 \cdot (-\sqrt{2}) - 3 \cdot 0 + 0 \cdot 0 \right] = -1.5 \sqrt{2} \]

\[ W_\psi (1,1) = \frac{1}{2} \left[ 1 \cdot 0 + 4 \cdot 0 - 3 \cdot \sqrt{2} + 0 \cdot (-\sqrt{2}) \right] = -1.5 \sqrt{2} \]
\[ f(x) = \frac{1}{2} \left[ W_\varphi (0,0) \varphi_{0,0} (x) + W_\psi (0,0) \psi_{0,0} (x) + W_\psi (1,0) \psi_{1,0} (x) + W_\psi (1,1) \psi_{1,1} (x) \right] \]

For \( x=0,1,2,3 \). If \( x=0 \), for instance,

\[ f(0) = \frac{1}{2} \left[ 1 \cdot 1 + 4 \cdot 1 - 1.5\sqrt{2} \cdot (\sqrt{2}) - 1.5\sqrt{2} \cdot 0 \right] = 1 \]
- The Wavelet Series Expansions
- The Discrete Wavelet Transform
- The Continuous Wavelet Transform
The continuous wavelet transform of a continuous, square-integrable function, \( f(x) \), relative to a real-valued wavelet, \( \psi(x) \), is

\[
W_\psi(s, \tau) = \int_{-\infty}^{\infty} f(x) \psi_{s,\tau}(x) dx
\]

Where

\[
\psi_{s,\tau}(x) = \frac{1}{\sqrt{s}} \psi \left( \frac{x - \tau}{s} \right)
\]

And \( s \) and \( \tau \) are called scale and translation parameters.
Given $W_\psi(s, \tau)$, $f(x)$ can be obtained using the inverse continuous wavelet transform

$$f(x) = \frac{1}{C_\psi} \int_{-\infty}^{\infty} \int_{0}^{\infty} W_\psi(s, \tau) \frac{\psi_{s,\tau}(x)}{s^2} d\tau ds$$

Where

$$C_\psi = \int_{-\infty}^{\infty} \left| \frac{\Psi(u)}{|u|} \right|^2 du$$

And $\Psi(u)$ is the Fourier transform of $\psi(x)$. 
The Mexican hat wavelet

\[ \psi(x) = \left( \frac{2}{\sqrt{3}} \pi^{-1/4} \right) (1 - x^2)e^{-x^2/2} \]
FIGURE 7.14 The continuous wavelet transform (c and d) and Fourier spectrum (b) of a continuous one-dimensional function (a).
• Background
• Multiresolution Expansions
• Wavelet Transforms in One Dimension
• Wavelet Transforms in Two Dimensions
In two dimensions, a two-dimensional scaling function, $\varphi(x, y)$, and three two-dimensional wavelet, $\psi^H(x, y)$, $\psi^V(x, y)$ and $\psi^V(x, y)$, are required.
Excluding products that produce one-dimensional results, like $\phi(x)\psi(x)$, the four remaining products produce the separable scaling function

$$\phi(x, y) = \phi(x)\phi(y)$$

And separable, “directionally sensitive” wavelets

$$\psi^H(x, y) = \psi(x)\phi(y)$$

$$\psi^V(x, y) = \phi(x)\psi(y)$$

$$\psi^D(x, y) = \psi(x)\psi(y)$$
The scaled and translated basis functions:

\[ \varphi_{j,m,n}(x, y) = 2^{j/2} \varphi(2^j x - m, 2^j y - n) \]

\[ \psi^i_{j,m,n}(x, y) = 2^{j/2} \psi(2^j x - m, 2^j y - n), \quad i = \{H, V, D\} \]
The discrete wavelet transform of function $f(x,y)$ of size $M*N$ is then

$$W_\varphi(j_0,m,n) = \frac{1}{\sqrt{MN}} \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x,y) \varphi_{j_0,m,n}(x,y)$$

$$W_\psi^i(j,m,n) = \frac{1}{\sqrt{MN}} \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x,y) \psi^i_{j,m,n}(x,y) \quad i = \{H,V,D\}$$
Given the $W_\varphi$ and $W_\psi^i$, $f(x,y)$ is obtained via the inverse discrete wavelet transform

$$f(x, y) = \frac{1}{\sqrt{MN}} \sum_{m} \sum_{n} W_\varphi(j_0, m, n) \varphi_{j_0, m, n}(x, y)$$

$$+ \frac{1}{\sqrt{MN}} \sum_{i=H,V,D} \sum_{j=j_0}^{\infty} \sum_{m} \sum_{n} W_\psi^i(j, m, n) \psi_{j, m, n}^i(x, y)$$
FIGURE 7.22 The two-dimensional fast wavelet transform: (a) the analysis filter bank; (b) the resulting decomposition; and (c) the synthesis filter bank.
FIGURE 7.23 A three-scale FWT.
Fourth-order symlets:
(a)–(b) decomposition filters;
(c)–(d) reconstruction filters;
(e) the one-dimensional wavelet; (f) the one-dimensional scaling function;
and (g) one of three two-dimensional wavelets, $\psi^H(x, y)$.
Fig. 7.24 (Con’t)
FIGURE 7.25
Modifying a DWT for edge detection: (a) and (c) two-scale decompositions with selected coefficients deleted; (b) and (d) the corresponding reconstructions.
FIGURE 7.26
Modifying a DWT for noise removal: (a) a noisy MRI of a human head; (b), (c) and (e) various reconstructions after thresholding the detail coefficients; (d) and (f) the information removed during the reconstruction of (e) and (e). (Original image courtesy Vanderbilt University Medical Center.)